1 Determination of NCCA in 2 dimensions

We want to consider number conserving CAs in 2 dimensions. There are different possibilities for the neighbourhood, both in size and in shape. To choose a criterium, I will determine all the rules for all the contigous neighbourhoods of sizes 3, 4 and 5, where the cells are said to be contigous if they share a side (**not** just a corner), and for each size, as much states as computationally possible will be considered. The size 2 is not interesting, for in that case the dynamics will be one dimensional, in diagonal, vertical or horizontal lines.

The first subsection (1.1) has the detail of the procedure which is also followed in the rest of this section:

- Listing of the possible shapes.
- Determination of a system of equations for each shape (necessary and sufficient condition).
- Listing the solutions, for each of the computationally feasible numbers of states (Q).
- Discarding of solutions, through considerations of **symmetry**, **conjugation**, and **real size** (the last point means the following: if the rule *ignores* some neighbours, then it may be a rule which has been found in a previous section).
- Description of the resulting set of rules.

1.1 Neighbourhood Size 3

The different possibilities of contigous shapes for neighbourhood size 3 are shown in figure 1. Only the shape on the left is interesting (the other is one dimensional).

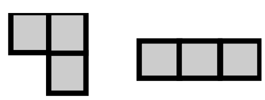

Fig. 1: Possible shapes for neighbourhoods of size 3

A first thing to notice (which is common to all the NCCAs), is that *all states must be quiescent*: $f(\begin{smallmatrix} i & i \\ & i \end{smallmatrix}) = i$ $\forall i = 0,\dots,Q-1$. In particular, $f(\begin{smallmatrix} 0 & 0 \\ & 0 \end{smallmatrix}) = 0$.

If a thorus is filled with zeroes, but for a region with the configuration $(\begin{smallmatrix} a & b \\ \cdot & c \end{smallmatrix})$), we see that any NC rule f must satisfy

$$a+b+c = f\begin{pmatrix} 0 & 0 \\ & b \end{pmatrix} + f\begin{pmatrix} b & 0 \\ & 0 \end{pmatrix} + f\begin{pmatrix} 0 & 0 \\ & a \end{pmatrix} + f\begin{pmatrix} a & b \\ & c \end{pmatrix} + f\begin{pmatrix} c & 0 \\ & 0 \end{pmatrix} + f\begin{pmatrix} 0 & a \\ & 0 \end{pmatrix} + f\begin{pmatrix} 0 & c \\ & 0 \end{pmatrix}$$

Putting isolated a and c on the thorus, we have

$$a+c = f\begin{pmatrix} a & 0 \\ & 0 \end{pmatrix} + f\begin{pmatrix} 0 & a \\ & 0 \end{pmatrix} + f\begin{pmatrix} 0 & 0 \\ & a \end{pmatrix} + f\begin{pmatrix} c & 0 \\ & 0 \end{pmatrix} + f\begin{pmatrix} 0 & c \\ & 0 \end{pmatrix} + f\begin{pmatrix} 0 & 0 \\ & c \end{pmatrix}$$

which may be sustracted from the first equation, obtaining

$$b = f\begin{pmatrix} a & b \\ & c \end{pmatrix} + \left[f\begin{pmatrix} b & 0 \\ & 0 \end{pmatrix} - f\begin{pmatrix} a & 0 \\ & 0 \end{pmatrix} \right] + \left[f\begin{pmatrix} 0 & 0 \\ & b \end{pmatrix} - f\begin{pmatrix} 0 & 0 \\ & c \end{pmatrix} \right]$$

Now, this is a necessary *and sufficient* condition for f to be number conserving, since if it verifies the equation, then we can add on the two sides, over all the positions of the thorus, and the sum of each of the terms in brackets will cancel, leaving only the sum of the states at one moment (on the left) and at the next moment (on the right).

Q	Rules	New rules
2	3	0
3	9	0
4	36	4
5	195	29

Tab. 1: Number of new rules with neighbourhood of size 3

We can determine all the number conserving rules f for these shapes, for a given number of states Q, by listing the solutions of the respective systems of equations, obtained in each case by replacing all the possible $a,b,c \in \{0,\dots,Q-1\}$.

The number of solutions for $Q = 2,3,4,5$ is shown in table 1.1. Some solutions can be discarded: first of all, the ones that ignore one of the positions in the neighbourhood, since the dynamics will be one dimensional. We can also discard rules according to conjugation, where the conjugated rule of f is defined as $f'(a,b,c) = Q-1-f(Q-1-a,Q-1-b,Q-1-c)$. And finally, we can discard one of each pair of symmetric rules. What is left is what I call "new rules".

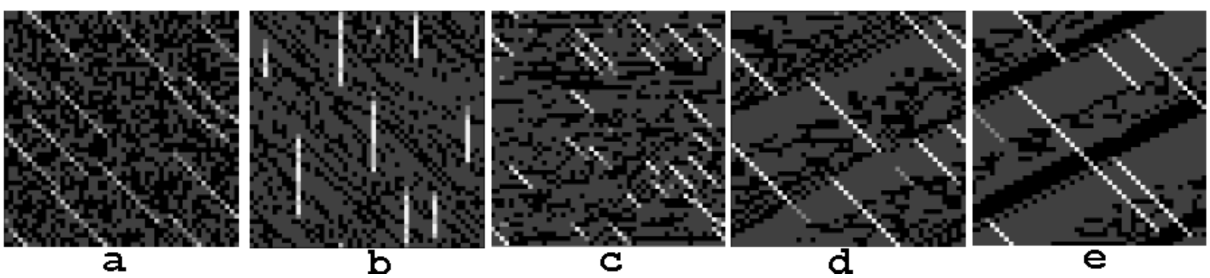

Fig. 2: Some rules, for neighbourhood size 3

Description of the rules

In most of the rules found here, the behavior after some number of iterations can be summarized as follows: at low density (20%), there is a nearly constant set of ones fixed or shifting on a background of zeroes. For high density (80%), $(Q-2)$'s shift or remain constant on a background of $(Q-1)$'s. In between, there works some cancelation law, that can lead to mixed shifts, fixed states or to caotic aspect, on middle density (50%). For some rules, some lines with 0/1 can remain at high density, or lines of

1	0011112200111122001111220011112211222233112222331122223311222233
2	0001000111121111211112111122223222311121111222232223111211112222322232223
3	000111121111211112000111121111211112111122223222322231112222322232223
4	0011001111221122112211222233223300110011112211221122112222332233

Tab. 2: New rules with neighbourhood of size 3, $Q = 4$

1	00011111221112211122222330001111122111221112222233111222223322233222333334411122222332223322233333441112222233222332223333344
2	00011000110001100011111221112211122111221112222233222332223322233222333334422233222332223322233333442223322233222332223333344
3	00011111220001100011111220001111122000110001111122222333334422233222333334422233333442223322233333442223333344222332223333344
4	00011000112223311122222331112211122333442223333344000110001122233111222223311122111223334422233333441112211122333442223333344
5	00011111222223311122222330001111122222331112222233000111112222233111222223311122222333334422233333441112222233333442223333344
6	00011000111112200011111221112211122222331112222233111221112222233111222223322233222333334422233333442223322233333442223333344
7	00011111221112200011111220001111122111220001111122111222223322233111222223322233333443334422233333442223333344333442223333344
8	00022000221113311133111331113311133222442224422244111331113322244222442224400022000221113311133111331113311133222442224422244
9	00022111331113311133111330002211133111331113311133111332224422244222442224400022111331113311133111331113322244222442224422244
10	00122001220012211233112331123311233112332234422344112331123311233223442234400122001220012211233112331123311233112332234422344
11	00001000011111222223222231111211112222233333433334111121111222223333343333411112111122222333334333341111211112222233333433334
12	00001111121111222223222230000111112111122222322223111122222322223333343333411112222232222333334333341111222223222233333433334
13	00001000010000111112111121111211112111122222322223222232222322223333343333422223222232222333334333342222322223222233333433334
14	00001111120000111112111120000111112000011111211112222233333422223333343333422223333342222333334333342222333334222233333433334
15	00001000012222322223222231111211112333343333433334000010000122223222232222311112111123333433334333341111211112333343333433334
16	00001111122222322223222230000111112222232222322223000011111222223222232222311112222233333433334333341111222223333343333433334
17	00001000011111211112111121111211112222232222322223111121111222223222232222322223222233333433334333342222322223333343333433334
18	00001111121111211112111120000111112111121111211112111122222322223222232222322223333343333433334333342222333334333343333433334
19	00011000111112222233222331112211122222333334433344111221112222233333443334400011000111112222233222331112211122222333334433344
20	00011111221112222233222330001111122111222223322233111222223322233333443334400011111221112222233222331112222233222333334433344
21	00011000110001111122111221112211122111222223322233222332223322233333443334411122111221112222233222332223322233222333334433344
22	00011111220001111122111220001111122000111112211122222333334422233333443334411122222331112222233222332223333344222333334433344
23	00011000112223322233222331112211122333443334433344000110001122233222332223300011000112223322233222331112211122333443334433344
24	00011000111112211122111221112211122222332223322233111221112222233222332223311122111222223322233222332223322233333443334433344
25	00011111221112211122111220001111122111221112211122111222223322233222332223311122222332223322233222332223333344333443334433344
26	00101001011121222323223231121211212223233343433434001010010111212223232232311212112122232333434334341121211212223233343433434
27	00101001010010111212112121121211212112122232322323112121121211212223232232322323223232232333434334342232322323223233343433434
28	00101112120010111212112120010111212001011121211212112122232311212223232232322323334342232333434334342232333434223233343433434
29	00101001011121211212112121121211212223232232322323001010010111212112121121211212223232232333434334342232322323334343343433434

Tab. 3: New rules with neighbourhood of size 3, $Q = 5$

$(Q-1),(Q-2)$ remain at low ones. Another typical behavior is of diagonal lines moving on a nearly constant background (in figure 2a, the lines are moving up and to the right). Another behavior (rule 4 for $Q=4$, 10 for $Q=5$; figure 2b) has vertical stripes of lower and higher states. Some exceptional behaviors are those of rule 7, $Q=5$, where the diagonal line "combs" the states to build a cluster (different stages in figure 2c-d-e), and rule 23, $Q=5$, where the interaction of vertical and horizontal shifts does not cause a cancelation, but a "turning", with nice effects (try density 70%, for instance).

In some cases, in the steady state, it is possible to interpret some transition as a movement of a particle in some direction. But, unfortunately, this is very, very far from actually interpreting the whole rules in terms of particles. More about this, later.

1.2 Neighbourhood Size 4

The possible shapes are shown in figure 3. Notice that with the "relaxed" definition of contigous cells, 17 other shapes would be added (I don't understand how or why I missed 9 of them in my original report). Since number 5 is one dimensional, we have four shapes to deal with.

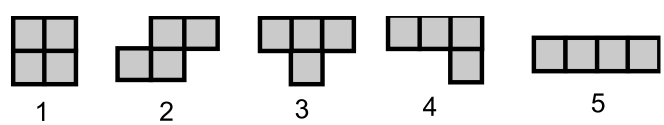

Fig. 3: Possible shapes for neighbourhoods of size 4

Shape 1
As before, we fill a thorus but for "one neighbourhood". Adding the states before and after applying

1	000000011001001012101101112111111011112112012212212112111222122001112012101212112
2	000000011001001012111111122111111011112112012222222122111111122001001012111111122
3	000000011001001012001001012111111011112112012112112012222222122112112012112112012
4	0000000110010010120110110221111110111121120121221220222222111122112001012122011022
5	000000011011011022101101112111000011122011022212101112111222122011122022101212112
6	000000011011011022111111122111000011122011022222111122111111122011011022111111122
7	000000011011011022001001012111000011122011022112001012222222122122122022112112012
8	0000000110110110220110110221110000111220110221220110222222111122122011022122011022
9	000000121101101222001001122000111121101212222001112122111111121101101111112112122
10	000000111001001112101101212111111111112112112212212212000111111001112112101212212
11	000000011101101112111111122000111011101212112111222122111111122101101112111111122
12	0000001110010011121111111222111111111121121122222222222000000111001001112111111222
13	000000011101101112011011022000111011101212112011122022222111122212101112122011022
14	000000111001001112011011122111111111112112112122122122111000111112001112122011122
15	00000000000100100111111111111111110001121120012222221111111111111121121122222222222
16	00000000001001001011111111111111000111121010121222111222111111111211211212222222222
17	000000111011011112210110121211100011112201112221210121200011111011122122101212212
18	000000011111111122111111122000000011111111122111111122111111122111111122111111122
19	0000001110110111122111111222111000111122011122222111222000000111011011122111111222
20	0000011110110121221111112222111001111122012122111001111000001111011012122111112222
21	00000011101101111220010011121110001111220111221120011121111111111122122122112112112
22	000000011111111122011011022000000011111111122011011022222111122222111122122011022
23	0000001110110111122011011122111000111122011122122011122111000111122011122122011122
24	000000111101101212111111222000111111101212212111222222000000111101101212111111222
25	000000111101101212011011122000111111101212212011122122111000111212101212122011122
26	000001101010011111111112212111001212121011222111001212000001101121122222111112212
27	000000211011011222001001212111000211122011222112001212000000100122122222112112212
28	000000111111111222011011122000000111111111222011011122111000111222111222122011122
29	000010111111121222011021122000010111000010111011021122111010111222121222122021122

Tab. 4: New rules with neighbourhood of size 4, $Q=3$, Shape 1

the rule, we have the following condition for the rule to be number conserving:

$$a+b+c+d = f\begin{pmatrix}0&0\\0&a\end{pmatrix}+f\begin{pmatrix}0&0\\a&b\end{pmatrix}+f\begin{pmatrix}0&0\\b&0\end{pmatrix}+f\begin{pmatrix}0&a\\0&c\end{pmatrix}+f\begin{pmatrix}a&b\\c&d\end{pmatrix}+f\begin{pmatrix}b&0\\d&0\end{pmatrix}+f\begin{pmatrix}0&c\\0&0\end{pmatrix}+f\begin{pmatrix}c&d\\0&0\end{pmatrix}+f\begin{pmatrix}d&0\\0&0\end{pmatrix}$$

Now, letting some of the variables be zero, we get the following equations:

$$\begin{aligned}
a+b &= f\begin{pmatrix}0&0\\0&a\end{pmatrix}+f\begin{pmatrix}0&0\\a&b\end{pmatrix}+f\begin{pmatrix}0&0\\b&0\end{pmatrix}+f\begin{pmatrix}0&a\\0&0\end{pmatrix}+f\begin{pmatrix}a&b\\0&0\end{pmatrix}+f\begin{pmatrix}b&0\\0&0\end{pmatrix}\\
a+c &= f\begin{pmatrix}0&0\\0&a\end{pmatrix}+f\begin{pmatrix}0&0\\a&0\end{pmatrix}+f\begin{pmatrix}0&a\\0&c\end{pmatrix}+f\begin{pmatrix}a&0\\c&0\end{pmatrix}+f\begin{pmatrix}0&c\\0&0\end{pmatrix}+f\begin{pmatrix}c&0\\0&0\end{pmatrix}\\
a &= f\begin{pmatrix}0&0\\0&a\end{pmatrix}+f\begin{pmatrix}0&0\\a&0\end{pmatrix}+f\begin{pmatrix}0&a\\0&0\end{pmatrix}+f\begin{pmatrix}a&0\\0&0\end{pmatrix}
\end{aligned}$$

Combining them, we get a necessary and sufficient condition:

$$d = f\begin{pmatrix}a&b\\c&d\end{pmatrix}+\left[f\begin{pmatrix}b&0\\d&0\end{pmatrix}-f\begin{pmatrix}a&0\\c&0\end{pmatrix}\right]+\left[f\begin{pmatrix}c&d\\0&0\end{pmatrix}-f\begin{pmatrix}a&b\\0&0\end{pmatrix}\right]+\left[f\begin{pmatrix}d&0\\0&0\end{pmatrix}-f\begin{pmatrix}b&0\\0&0\end{pmatrix}\right]+\left[f\begin{pmatrix}a&0\\0&0\end{pmatrix}-f\begin{pmatrix}c&0\\0&0\end{pmatrix}\right]$$

With (a,b,c,d) ranging over $\{0,\ldots,Q-1\}^4$, we obtain for each Q an equation system, whose solutions I listed for $Q=2$ and $Q=3$. For $Q=4$, there are already 27 degrees of freedom, and the listing of the solutions becomes computationally too hard. 4 rules were found for $Q=2$ (identity and shifts), and 338 were found for $Q=3$.

As before, I discarded rules according to conjugation, symmetry, and non-dependance from some cell of the neighbourhood (for not depending on any of the cells would restrict the rule to the cases of the previous section). 29 were the “new rules” found after this discarding, for $Q=3$, and are shown in table 1.2

Shape 2

30	000000000100100100111111111000111111100211211111222222111111111211211211222222222
31	000000011111111112221121122200000000111111111222112112220001111220001111122100211222
32	000000111001001112001001112111111111111121121121121121121111111111121121121121121112
33	000000121001001122001001122111111112111211212211211212211111112100100101111212112122
34	000000211001001212001001212111111211111211221211211221200000010011211221211212112212
35	000000221001001222001001222111111221112112222112112222000000110001001111112112222
36	000001001000001001011012012111111211211111211201101201222211211222211211212201 2012
37	000001001000001001111112112111111211211111211211111211211111211211111211211111 2112
38	000001001000001001121122122111111211211111211212112212211100111211100111212101 1122
39	000001001000001001211121221211111211211111211221121221200011211200011211210021 2212
40	000001001000001001221222222111111211211111211222122222000001112000001112110111222
41	000001001010011011111111211211100111212101112211100111211111211212112212211111 2112
42	000001001010011011112112212211100111212101112212101112211100111212101112212101 1122
43	000001001010011011121121221211100111212101112221110121200011211201012212210021 2212
44	000001001010011011221222222111001112121011122221111222000001112010011122110111222
45	000001001100101101121122122000112112100212212010122122111001112211101212121011122
46	000001001100101101221222222000112112100212212110222222000001112100101212110111222
47	000001001110111111121122122000001112110111222010011122111001112221111222121011122
48	000001001110111111221222222000001112110111222110111222000001112110111222110111222
49	000001011000001011011012022111112122111112122011012022222112122111001011122012022
50	000001011000001011111112122111112122111112122111112122111112122000001011111112122
51	000001011011012022111112122111001011122012022111001011111112122011012022111112122
52	000001011100101111111112122000112122100212222000112122111112122100101111111112122
53	000001011111112122111112122000001011111112122000001011111112122111112122111112122
54	000001011111112122211212222000001011111112122100101111000112122000112122100212222
55	000001101000001101011012112111111221211111221201101211211100110122211221212201 2112
56	000001101000001101111112212111111221211111221211111221200000110111111221211111 2212
57	000001111000001111000001111111111222211111222200000111111111222211111222211111 2222
58	000001111000001111011012122111112222111112222011012122111001111111001111122012122
59	000001111000001111111111222211111222211111222211111222200000111100000111111111 2222
60	000001111111111222211111222200000111111111222200000111100000111111111222211111 2222
61	000010011111121122011021022000010011000010011011021022222121122222121122122021022
62	000010011111121122111121122000010011000010011111121122111121122111121122111121122
63	000010011111121122211221222000010011000010011211221222000121122000121122100221222
64	000010111000010111000010111111121222000010111111121222111121222111121222111121222
65	000010111001011112001011112111121111001011001112122112111121111112122112112122112
66	000010111001011112101111212111121111001011001212222212000121111001122112101222212
67	000010111111121222111121222000010111000010111111121222000010111111121222111121222
68	000010121001011122001011122111121121001011011112122122111121121001011011112122122
69	000010211001011212001011212111121211001011101112122212000010100112122212112122212
70	000011001000011001011022012111122112000011001011022012222122112222122112122022012
71	000011001000011001111122112111122112000011001111122112111122112111122112111122112
72	000011001000011001211222212111122112000011001211222212000122112000122112100222212
73	000011011000011011011022022111122122000011011011022022222122122111011011122022022
74	000011011000011011111122122111122122000011011111122122111122122000011011111122122
75	000011011111122122111122122000011011000011011000011011111122122111122122111122122
76	000011101000011101011022112111122212000011101011022112111011101222122212122022112
77	000011101000011101111122212111122212000011101111122212000011101111122212111122212
78	000011111000011111011022122111122222000011111011022122111011111111011111122022122
79	000011111001012112011022122111122111001012001011022011111011111112012112122022122
80	000100111000100111000100111000100111111211222111211222111211222111211222111211222
81	000100121001101122001101122000100010112212122112212122111211121001101011112212122
82	000101011000101011011112022000101011111212122011112022222212122111101011122112022
83	000101011000101011111212122000101011111212122111212122111212122000101011111212122
84	000101111000101111011111212200010111111121222201111212211110111111110111111221 2122
85	000110121001111122001111122000110010001111011112222122111221121001111011112222122
86	000111011000111011011122022000111011000111011011122022222222122111111011122122022
87	000111011000111011111122212200011101100011101111122212211122212200011101111122 2122
88	000111011010121021011122022000000011010010021011011022222222122121121021122122022
89	000111011010121021111222122000000011010010021111111122111222122010121021111222122
90	000111111000111111011122122000111111000111111011122122111111111111111111122122122
91	000111111001112112011122122000111000001112001011122011111111111121121121122122122
92	000111111010121121011122122000000111010010121011011122111111111121121121122122122
93	000111111011122122011122122000000000011011011011011111111111122122122122122122
94	001000011102101112001000011001111011102212112112222122112111122102101112112111122
95	001000011112111122001000011001000011112111122112111122112111122112111122112111122
96	001001001001001001011011011112112112112112112122122122112001112112001112122011122
97	001001001001001001101101101112112112112112122122122120011121120011121121012122 12
98	001001001011011011001001001112001112122011122112001112112112112122122122112112112
99	001001001011011011011011011112001112122011122122011122112001112122011122122011122
100	001001001011011011101101101112001112122011122212101212001112112011122122101212212
101	001001001101101101011011011001112112101212212011122122112001112212101212122011122
102	001001001111111111011011011001001112111111222011011122112001112222111222122011122
103	001001011001001011001001011111211212211211212211211212211211212200100101111212112122
104	001001011101101111001001011001112122101212222001112122112112122101101111112112122
105	001001011112112122001001011001001011112112122001001011112112122112112122112112122
106	001001101001001101001001101112112212112112212112112212001001101112112212112112212
107	001001101011011111001001101112001212122011222112001212001001101122122222112112212
108	001010011112121122001010011001010011001010011112121122112121122112121122112121122
109	001011001001011001001011001112122112001011001112122112112122112112122112112122112
110	001011001001011001101111101112122112001011001212222212001122112001122112101222212
111	001011001111121111011021011001011112000010111011021122112011112222121222122021122
112	001011101001011101001011101112122212001011101112122212001011101112122212112122212

Tab. 5: New rules with neighbourhood of size 4, $Q = 3$, Shape 2. The first 29 rules are not shown, since they coincide with those on table 1.2.

1	001001001001112112001112112112112112001112112001112112112112112001112112001112112
2	001001001112112112001112112001001001112112112001112112112112112112112112001112112
3	001001001001112112112112112112112112112112001112112112112001001001112112112112112

Tab. 6: New rules with neighbourhood of size 4, $Q=3$, Shape 3

1	001001001001112112001112112112112112001112112001112112112112112001112112001112112
2	001001001112112112001112112001001001112112112001112112112112112112112112001112112
3	001001001001112112112112112112112112112112001112112112112001001001112112112112112
4	001001112112112112001001001001112112112112112112112112001001112112112112112112112
5	001112001001001001112112112001112001112112112112112112001112001112112112112112112
6	001112112001001001001001001112112112112112112112112112001112112112112112112112112

Tab. 7: New rules with neighbourhood of size 4, $Q=3$, Shape 4

The equation is the same as before, and therefore we start with the same solutions. But now there is only one symmetry we can use to discard rules. I also discarded all the rules that ignore any of the 4 positions; ignoring a or d in $\begin{smallmatrix} & a & b \\ c & d & \end{smallmatrix}$ will produce new rules, but they will be taken into account with shape 4. The number of new rules, then, is 0 (again) for $Q=2$, and 112 for $Q=3$.

Shape 3

Now the necessary condition is

$$\begin{aligned} a+b+c+d = \; & f\begin{pmatrix} 0\;0\;0 \\ a \end{pmatrix} + f\begin{pmatrix} 0\;0\;0 \\ b \end{pmatrix} + f\begin{pmatrix} 0\;0\;0 \\ c \end{pmatrix} + f\begin{pmatrix} 0\;0\;a \\ 0 \end{pmatrix} + f\begin{pmatrix} 0\;a\;b \\ 0 \end{pmatrix} + f\begin{pmatrix} a\;b\;c \\ d \end{pmatrix} \\ & + f\begin{pmatrix} b\;c\;0 \\ 0 \end{pmatrix} + f\begin{pmatrix} c\;0\;0 \\ 0 \end{pmatrix} + f\begin{pmatrix} 0\;0\;d \\ 0 \end{pmatrix} + f\begin{pmatrix} 0\;d\;0 \\ 0 \end{pmatrix} + f\begin{pmatrix} d\;0\;0 \\ 0 \end{pmatrix} \end{aligned} \tag{1}$$

which in turns leads to

$$a+b = f\begin{pmatrix} 0\;0\;0 \\ a \end{pmatrix} + f\begin{pmatrix} 0\;0\;0 \\ b \end{pmatrix} + f\begin{pmatrix} 0\;0\;a \\ 0 \end{pmatrix} + f\begin{pmatrix} 0\;a\;b \\ 0 \end{pmatrix} + f\begin{pmatrix} a\;b\;0 \\ 0 \end{pmatrix} + f\begin{pmatrix} b\;0\;0 \\ 0 \end{pmatrix} \tag{2}$$

$$d = f\begin{pmatrix} 0\;0\;d \\ 0 \end{pmatrix} + f\begin{pmatrix} 0\;d\;0 \\ 0 \end{pmatrix} + f\begin{pmatrix} d\;0\;0 \\ 0 \end{pmatrix} + f\begin{pmatrix} 0\;0\;0 \\ d \end{pmatrix} \tag{3}$$

and with (1)-(2)-(3) we get the necessary and sufficient condition

$$c = f\begin{pmatrix} a\;b\;c \\ d \end{pmatrix} + \left[f\begin{pmatrix} 0\;0\;0 \\ c \end{pmatrix} - f\begin{pmatrix} 0\;0\;0 \\ d \end{pmatrix} \right] + \left[f\begin{pmatrix} b\;c\;0 \\ 0 \end{pmatrix} - f\begin{pmatrix} a\;b\;0 \\ 0 \end{pmatrix} \right] + \left[f\begin{pmatrix} c\;0\;0 \\ 0 \end{pmatrix} - f\begin{pmatrix} b\;0\;0 \\ 0 \end{pmatrix} \right]$$

We found 6 solutions for $Q=2$, and 163 for $Q=3$. Then I proceeded to discard all the rules that didn't consider a, c or d in $f\begin{pmatrix} a\;b\;c \\ d \end{pmatrix}$; ignoring b was allowed (for the resulting shape hasn't been considered before). Conjugation, and (horizontal) symmetry were also used to discard. No rule survived for $Q=2$, and only 3 did for $Q=3$.

For this equation, *it was* possible to list the solutions for $Q=4$, since there were "only" 18 degrees of freedom. I don't list them here, nor did I include them in the applet, but I have the list; they are a few thousands.

Shape 4

Equation and solutions are the same as for shape 3. Now we can't use symmetry to discard. With respect to the names $\begin{smallmatrix} a & b & c \\ & d & \end{smallmatrix}$, I discarded the rules that ignored a, d, or that ignored $\{b,c\}$ (for, in that case, the dynamics will be one dimensional, on one or two lines wrapped around the thorus). After discarding, no one rule was left for $Q=2$, and 6 did for $Q=3$.

Description of the rules

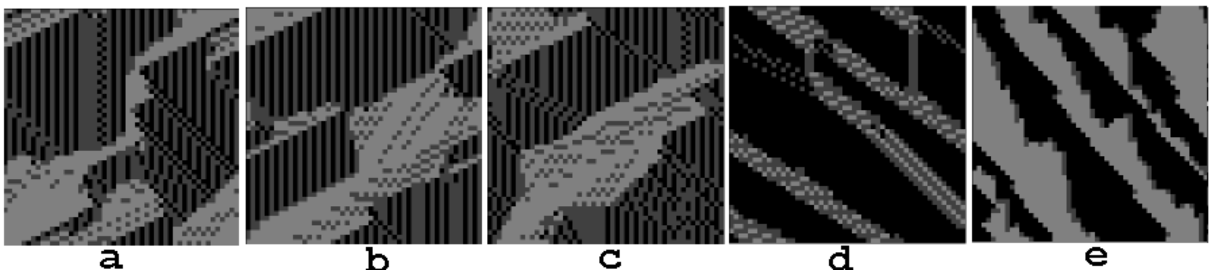

Fig. 4: Some rules, for neighbourhood size 4, shape 1

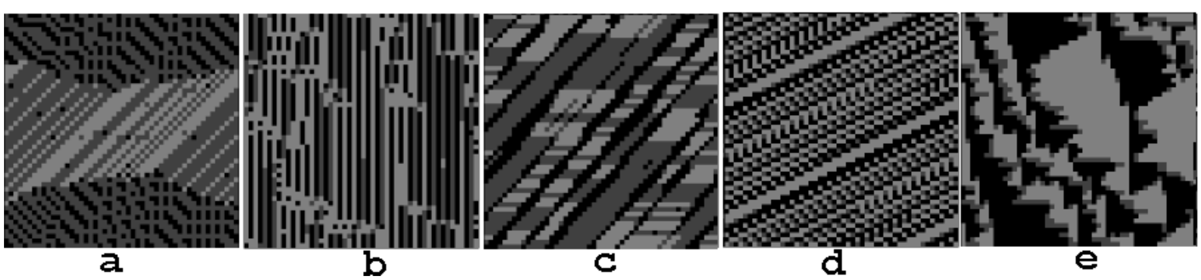

Fig. 5: Some rules, for neighbourhood size 4, shape 2

Q	Shape 1	Shape 2	Shape 3	Shape 4
2	$4 \to 0$	$4 \to 0$	$6 \to 0$	$6 \to 0$
3	$338 \to 29$	$338 \to 112$	$163 \to 3$	$163 \to 6$
4	?	?	$97983 \to 2247$	$97983 \to 4227$

Tab. 8: Number of rules and new rules for the different shapes, with neighbourhood size 4.

There are too much rules here; let's consider the rules with $Q = 3$. Starting with shape 1, some rules are found to be similar to those of the previous section (orthogonal shifts, canceling with different effects for middle densities). The new fenomenon is the appeareance of cluster, which may have a fixed form, or change it all the time (figure 4a-b-c are different moments of a same cluster, for rule 2), shifting in some direction, usually with speed 1/2 (in figure 4d, which corresponds to rule 6, the stripe shifts down with that speed). Figure 4e corresponds to rule 21. Some typical cases for shape 2 are given in figure 5, for rules 14, 49, 57, 63, and 71 (a number of rules gives patterns that look random, and are not worth printing, though a movement can be perceived while looking at them running). The few rules found for shapes 3 and 4 are all of the kind "superposition of shifts".

1.3 Neighbourhood Size 5

The possible shapes are shown in figure 6. The following equations give the necessary and sufficient conditions for the respective shapes (in some cases, one equation works for two shapes).

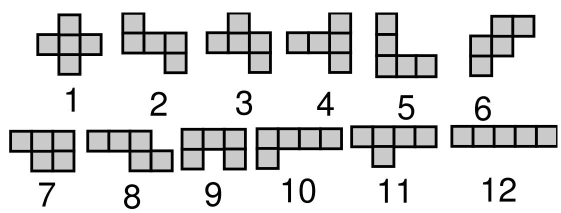

Fig. 6: Possible shapes for neighbourhoods of size 5

For shapes 1 and 2:

$$f\left(\begin{smallmatrix} & a & \\ b & c & d \\ & e & \end{smallmatrix}\right) = c + \left[f\left(\begin{smallmatrix} & 0 & \\ b & c & 0 \\ & 0 & \end{smallmatrix}\right) - f\left(\begin{smallmatrix} & 0 & \\ c & d & 0 \\ & 0 & \end{smallmatrix}\right)\right] + \left[f\left(\begin{smallmatrix} & a & \\ 0 & c & 0 \\ & 0 & \end{smallmatrix}\right) - f\left(\begin{smallmatrix} & c & \\ 0 & e & 0 \\ & 0 & \end{smallmatrix}\right)\right]$$

$$+ \left[f\left(\begin{smallmatrix} & a & \\ 0 & 0 & d \\ & 0 & \end{smallmatrix}\right) - f\left(\begin{smallmatrix} & b & \\ 0 & 0 & e \\ & 0 & \end{smallmatrix}\right)\right] + \left[f\left(\begin{smallmatrix} & 0 & \\ 0 & 0 & d \\ & e & \end{smallmatrix}\right) - f\left(\begin{smallmatrix} & 0 & \\ 0 & 0 & a \\ & b & \end{smallmatrix}\right)\right] + \left[f\left(\begin{smallmatrix} & b & \\ 0 & 0 & 0 \\ & 0 & \end{smallmatrix}\right) - f\left(\begin{smallmatrix} & a & \\ 0 & 0 & 0 \\ & 0 & \end{smallmatrix}\right)\right] + \left[f\left(\begin{smallmatrix} & 0 & \\ 0 & 0 & 0 \\ & b & \end{smallmatrix}\right) - f\left(\begin{smallmatrix} & 0 & \\ 0 & 0 & 0 \\ & c & \end{smallmatrix}\right)\right]$$

$$+ \left[f\left(\begin{smallmatrix} & 0 & \\ 0 & 0 & e \\ & 0 & \end{smallmatrix}\right) - f\left(\begin{smallmatrix} & 0 & \\ 0 & 0 & d \\ & 0 & \end{smallmatrix}\right)\right] + \left[f\left(\begin{smallmatrix} & 0 & \\ 0 & 0 & a \\ & 0 & \end{smallmatrix}\right) - f\left(\begin{smallmatrix} & 0 & \\ 0 & 0 & c \\ & 0 & \end{smallmatrix}\right)\right] + \left[f\left(\begin{smallmatrix} & 0 & \\ 0 & e & 0 \\ & 0 & \end{smallmatrix}\right) + f\left(\begin{smallmatrix} & 0 & \\ 0 & d & 0 \\ & 0 & \end{smallmatrix}\right) - 2f\left(\begin{smallmatrix} & 0 & \\ 0 & c & 0 \\ & 0 & \end{smallmatrix}\right)\right]$$

For shape 3:

$$f\left(\begin{smallmatrix} & a & \\ b & c & d \\ & & e \end{smallmatrix}\right) = c + \left[f\left(\begin{smallmatrix} & 0 & \\ b & c & 0 \\ & & 0 \end{smallmatrix}\right) - f\left(\begin{smallmatrix} & 0 & \\ c & d & 0 \\ & & 0 \end{smallmatrix}\right)\right] + \left[f\left(\begin{smallmatrix} & a & \\ 0 & 0 & d \\ & & 0 \end{smallmatrix}\right) - f\left(\begin{smallmatrix} & c & \\ 0 & 0 & e \\ & & 0 \end{smallmatrix}\right)\right] + \left[f\left(\begin{smallmatrix} & 0 & \\ 0 & 0 & d \\ & & e \end{smallmatrix}\right) - f\left(\begin{smallmatrix} & 0 & \\ 0 & 0 & a \\ & & c \end{smallmatrix}\right)\right]$$

$$+ \left[f\left(\begin{smallmatrix} & 0 & \\ 0 & 0 & a \\ & & 0 \end{smallmatrix}\right) - f\left(\begin{smallmatrix} & 0 & \\ 0 & 0 & d \\ & & 0 \end{smallmatrix}\right)\right] + \left[f\left(\begin{smallmatrix} & 0 & \\ 0 & d & 0 \\ & & 0 \end{smallmatrix}\right) - f\left(\begin{smallmatrix} & 0 & \\ 0 & c & 0 \\ & & 0 \end{smallmatrix}\right)\right] + \left[f\left(\begin{smallmatrix} & 0 & \\ 0 & 0 & e \\ & & 0 \end{smallmatrix}\right) - f\left(\begin{smallmatrix} & 0 & \\ 0 & 0 & c \\ & & 0 \end{smallmatrix}\right)\right]$$

Q	1	2	3	4	5	6	7	8	9	10	11
2	$7 \to 0$	$7 \to 0$	$7 \to 0$	$9 \to 0$	$9 \to 0$	$11 \to 1$	$11 \to 2$	$11 \to 2$	$7 \to 0$	$23 \to 0$	$23 \to 0$
3	$492 \to 62$	$492 \to 149$	$690 \to 97$	$471 \to 36$	$415 \to 23$	?→?	?→?	?→?	$630 \to 27$	?→?	?→?

Tab. 9: Number of rules and new rules for the different shapes, with neighbourhood size 5.

For shape 4:

$$f\begin{pmatrix} & & a \\ b & c & d \\ & & e \end{pmatrix} = a + \left[f\begin{pmatrix} & & 0 \\ b & c & 0 \\ & & 0 \end{pmatrix} - f\begin{pmatrix} & & 0 \\ c & d & 0 \\ & & 0 \end{pmatrix} \right] + \left[f\begin{pmatrix} & & 0 \\ 0 & 0 & d \\ & & e \end{pmatrix} - f\begin{pmatrix} & & 0 \\ 0 & 0 & a \\ & & d \end{pmatrix} \right]$$

$$+ \left[f\begin{pmatrix} & & 0 \\ 0 & d & 0 \\ & & 0 \end{pmatrix} - f\begin{pmatrix} & & 0 \\ 0 & a & 0 \\ & & 0 \end{pmatrix} \right] + \left[f\begin{pmatrix} & & 0 \\ 0 & 0 & 0 \\ & & d \end{pmatrix} - f\begin{pmatrix} & & 0 \\ 0 & 0 & 0 \\ & & a \end{pmatrix} \right] + \left[f\begin{pmatrix} & & 0 \\ c & 0 & 0 \\ & & 0 \end{pmatrix} - f\begin{pmatrix} & & 0 \\ a & 0 & 0 \\ & & 0 \end{pmatrix} \right]$$

For shape 5:

$$f\begin{pmatrix} a & b & c \\ & & d \\ & & e \end{pmatrix} = c + \left[f\begin{pmatrix} 0 & 0 & 0 \\ & & d \\ & & e \end{pmatrix} - f\begin{pmatrix} 0 & 0 & 0 \\ & & c \\ & & d \end{pmatrix} \right] + \left[f\begin{pmatrix} a & b & 0 \\ & & 0 \\ & & 0 \end{pmatrix} - f\begin{pmatrix} b & c & 0 \\ & & 0 \\ & & 0 \end{pmatrix} \right]$$

$$+ \left[f\begin{pmatrix} b & 0 & 0 \\ & & 0 \\ & & 0 \end{pmatrix} - f\begin{pmatrix} c & 0 & 0 \\ & & 0 \\ & & 0 \end{pmatrix} \right] + \left[f\begin{pmatrix} 0 & 0 & 0 \\ & & 0 \\ & & d \end{pmatrix} - f\begin{pmatrix} 0 & 0 & 0 \\ & & 0 \\ & & c \end{pmatrix} \right]$$

For shape 6:

$$f\begin{pmatrix} & a & b \\ c & d & \\ e & & \end{pmatrix} = e + \left[f\begin{pmatrix} & a & b \\ 0 & d & \\ 0 & & \end{pmatrix} - f\begin{pmatrix} & c & d \\ 0 & e & \\ 0 & & \end{pmatrix} \right] + \left[f\begin{pmatrix} & a & 0 \\ c & 0 & \\ 0 & & \end{pmatrix} - f\begin{pmatrix} & d & 0 \\ e & 0 & \\ 0 & & \end{pmatrix} \right]$$

$$+ \left[f\begin{pmatrix} & 0 & d \\ 0 & 0 & \\ 0 & & \end{pmatrix} - f\begin{pmatrix} & 0 & e \\ 0 & 0 & \\ 0 & & \end{pmatrix} \right] + \left[f\begin{pmatrix} & d & 0 \\ 0 & 0 & \\ 0 & & \end{pmatrix} - f\begin{pmatrix} & e & 0 \\ 0 & 0 & \\ 0 & & \end{pmatrix} \right] + \left[f\begin{pmatrix} & c & 0 \\ 0 & 0 & \\ 0 & & \end{pmatrix} - f\begin{pmatrix} & a & 0 \\ 0 & 0 & \\ 0 & & \end{pmatrix} \right]$$

For shapes 7 and 8:

$$f\begin{pmatrix} a & b & c \\ & d & e \end{pmatrix} = c + \left[f\begin{pmatrix} a & b & 0 \\ & d & 0 \end{pmatrix} - f\begin{pmatrix} b & c & 0 \\ & e & 0 \end{pmatrix} \right] + \left[f\begin{pmatrix} 0 & 0 & 0 \\ & d & e \end{pmatrix} - f\begin{pmatrix} 0 & 0 & 0 \\ & b & c \end{pmatrix} \right]$$

$$+ \left[f\begin{pmatrix} 0 & 0 & 0 \\ & b & 0 \end{pmatrix} - f\begin{pmatrix} 0 & 0 & 0 \\ & c & 0 \end{pmatrix} \right] + \left[f\begin{pmatrix} b & 0 & 0 \\ & 0 & 0 \end{pmatrix} - f\begin{pmatrix} c & 0 & 0 \\ & 0 & 0 \end{pmatrix} \right] + \left[f\begin{pmatrix} 0 & 0 & 0 \\ & e & 0 \end{pmatrix} - f\begin{pmatrix} 0 & 0 & 0 \\ & d & 0 \end{pmatrix} \right]$$

For shape 9:

$$f\begin{pmatrix} a & b & c \\ d & & e \end{pmatrix} = c + \left[f\begin{pmatrix} 0 & 0 & 0 \\ d & & e \end{pmatrix} - f\begin{pmatrix} 0 & 0 & 0 \\ a & & c \end{pmatrix} \right] + \left[f\begin{pmatrix} a & b & 0 \\ 0 & & 0 \end{pmatrix} - f\begin{pmatrix} b & c & 0 \\ 0 & & 0 \end{pmatrix} \right] + \left[f\begin{pmatrix} 0 & 0 & c \\ 0 & & e \end{pmatrix} - f\begin{pmatrix} 0 & 0 & a \\ 0 & & d \end{pmatrix} \right]$$

$$+ \left[f\begin{pmatrix} b & 0 & 0 \\ 0 & & 0 \end{pmatrix} - f\begin{pmatrix} c & 0 & 0 \\ 0 & & 0 \end{pmatrix} \right] + \left[f\begin{pmatrix} 0 & 0 & 0 \\ a & & 0 \end{pmatrix} - f\begin{pmatrix} 0 & 0 & 0 \\ c & & 0 \end{pmatrix} \right] + \left[f\begin{pmatrix} 0 & 0 & a \\ 0 & & 0 \end{pmatrix} - f\begin{pmatrix} 0 & 0 & c \\ 0 & & 0 \end{pmatrix} \right] + \left[f\begin{pmatrix} 0 & 0 & 0 \\ 0 & & d \end{pmatrix} - f\begin{pmatrix} 0 & 0 & 0 \\ 0 & & e \end{pmatrix} \right]$$

For shapes 10 and 11:

$$f\begin{pmatrix} a & b & c & d \\ e & & & \end{pmatrix} = d + \left[f\begin{pmatrix} a & b & c & 0 \\ 0 & & & \end{pmatrix} - f\begin{pmatrix} b & c & d & 0 \\ 0 & & & \end{pmatrix} \right] + \left[f\begin{pmatrix} b & c & 0 & 0 \\ 0 & & & \end{pmatrix} - f\begin{pmatrix} c & d & 0 & 0 \\ 0 & & & \end{pmatrix} \right]$$

$$+ \left[f\begin{pmatrix} c & 0 & 0 & 0 \\ 0 & & & \end{pmatrix} - f\begin{pmatrix} d & 0 & 0 & 0 \\ 0 & & & \end{pmatrix} \right] + \left[f\begin{pmatrix} 0 & 0 & 0 & 0 \\ e & & & \end{pmatrix} - f\begin{pmatrix} 0 & 0 & 0 & 0 \\ d & & & \end{pmatrix} \right]$$

Description of the rules

The rules found for $Q = 2$, for shapes 6, 7 and 8, are almost constant, but for some transitions that produce stripes of the kind shown in figure 7a. For $Q = 3$, the main novelty, in relation to the rules of the previous sections, is the appearance of cluster that remain stable, with activity in their borders (7b). Another new behaviour is the presence of shifts in different directions: in 7c, the "doted lines" above move to the right, while the triangle below moves to the left. In 7d, some lines shift in one direction and some in the other, and in 7e, the two block shift in opposite directions.

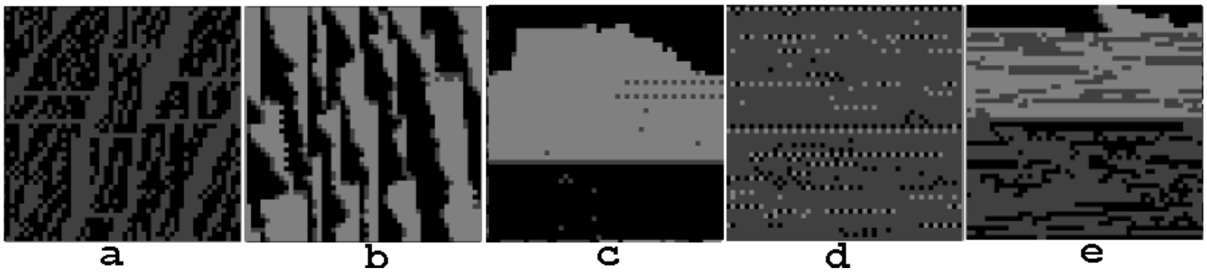

Fig. 7: Some rules, for neighbourhood size 5

1.4 General equation for rectangular neighbourhoods

Here I show how to write an equation of the style used before, for any rectangular neighbourhood, for any number of states.

Note that this implies the existence of a system of equations for any neighbourhood (of any shape): just take the equations for the rectangle that covers it, and add the equations to force the resulting rules to ignore the positions which are not part of the shape. This does **not** imply that for any shape, there should be a single expression which generates the whole system of equations, when the variables range over $0,\dots,Q-1$. Nevertheless, that was the case for all the shapes considered in the previous sections, and I guess that it will be the case for any shape.

(I think it has to do with a symmetry that arises when you write the initial, necessary condition: to write the terms, you move the neighbourhood over a zone of the same shape which is filled with the variables; for each position of the neighbourhood over that zone, there will be a position where neighbourhood and zone are in the same relative positions, but with the roles inverted; the only case where these symmetry doesn't exist, is when the neighbourhood is exactly on the zone, and that's the term you want to conserve. I don't see it clearly, but it must be something like that.)

Let's consider a neighbourhood of n rows and m columns. As before, we take a thorus filled with zeroes, but for a region with the matrix

$$A = \begin{bmatrix} a_{1,1} & \dots & a_{1,m} \\ \vdots & \ddots & \vdots \\ a_{n,1} & \dots & a_{n,m} \end{bmatrix}$$

Since we need to write a lot of matrices, which are all of them submatrices of A put in a corner of a matrix of size $n \times m$, the rest of which is filled with zeroes, we will use the notation $\left.\underline{\begin{matrix} 2,2;2,3 \\ n,2;n,7 \end{matrix}}\right|$, for instance, to represent the image of the matrix

$$\begin{bmatrix} 0 & \dots & \dots & \dots & \dots & 0 \\ \vdots & & & \vdots & & \vdots \\ 0 & \dots & 0 & 0 & \dots & 0 \\ 0 & \dots & 0 & a_{2,2} & \dots & a_{2,7} \\ \vdots & & \vdots & \vdots & & \vdots \\ 0 & \dots & 0 & a_{n,2} & \dots & a_{n,7} \end{bmatrix}$$

or $\left|\overline{\,n,3;n,m\,}\right.$, for the image of a matrix filled with zeroes but for the first row, which starts with $a_{n,3}, a_{n,4}, \dots, a_{n,m}$.

We start with the necessary condition:

$$
\sum_{i=1}^{n}\sum_{j=1}^{m} a_{i,j} = \begin{array}{ccccccccc}
 & \boxed{1,1} & + \cdots + & \boxed{1,1;1,m} & + \cdots + & \boxed{1,m} \\
+ & \boxed{\substack{1,1\\2,1}} & + \cdots + & \boxed{\substack{1,1;1,m\\2,1;2,m}} & + \cdots + & \boxed{\substack{1,m\\2,m}} \\
 & \vdots & \vdots & \vdots & \vdots & \vdots \\
+ & \boxed{\substack{1,1\\n,1}} & + \cdots + & \boxed{\substack{1,1;1,m\\n,1;n,m}} & + \cdots + & \boxed{\substack{1,m\\n,m}} \\
 & \vdots & \vdots & \vdots & \vdots & \vdots \\
+ & \boxed{\substack{n-1,1\\n,1}} & + \cdots + & \boxed{\substack{n-1,1;n-1,m\\n,1;n,m}} & + \cdots + & \boxed{\substack{n-1,m\\n,m}} \\
+ & \boxed{n,1} & + \cdots + & \boxed{n,1;n,m} & + \cdots + & \boxed{n,m}
\end{array} \tag{4}
$$

Now, taking $a_{1,j} = 0$ for all j, we have

$$
\sum_{i=2}^{n}\sum_{j=1}^{m} a_{i,j} = \begin{array}{ccccccccc}
 & \boxed{2,1} & + \cdots + & \boxed{2,1;1,m} & + \cdots + & \boxed{2,m} \\
+ & \boxed{\substack{2,1\\3,1}} & + \cdots + & \boxed{\substack{2,1;2,m\\3,1;3,m}} & + \cdots + & \boxed{\substack{2,m\\3,m}} \\
 & \vdots & \vdots & \vdots & \vdots & \vdots \\
+ & \boxed{\substack{2,1\\n,1}} & + \cdots + & \boxed{\substack{2,1;2,m\\n,1;n,m}} & + \cdots + & \boxed{\substack{2,m\\n,m}} \\
+ & \boxed{\substack{2,1\\n,1}} & + \cdots + & \boxed{\substack{2,1;2,m\\n,1;n,m}} & + \cdots + & \boxed{\substack{2,m\\n,m}} \\
 & \vdots & \vdots & \vdots & \vdots & \vdots \\
+ & \boxed{\substack{n-1,1\\n,1}} & + \cdots + & \boxed{\substack{n-1,1;n-1,m\\n,1;n,m}} & + \cdots + & \boxed{\substack{n-1,m\\n,m}} \\
+ & \boxed{n,1} & + \cdots + & \boxed{n,1;n,m} & + \cdots + & \boxed{n,m}
\end{array} \tag{5}
$$

Taking $a_{i,1} = 0$ for all i, we have

$$
\sum_{i=1}^{n}\sum_{j=2}^{m} a_{i,j} = \begin{array}{ccccccccccc}
 & \boxed{1,2} & + \cdots + & \boxed{1,2;1,m} & + & \boxed{1,2;1,m} & + \cdots + & \boxed{1,m} \\
+ & \boxed{\substack{1,2\\2,2}} & + \cdots + & \boxed{\substack{1,2;1,m\\2,2;2,m}} & + & \boxed{\substack{1,2;1,m\\2,2;2,m}} & + \cdots + & \boxed{\substack{1,m\\2,m}} \\
 & \vdots & \vdots & \vdots & & \vdots & \vdots & \vdots \\
+ & \boxed{\substack{1,2\\n,2}} & + \cdots + & \boxed{\substack{1,2;1,m\\n,2;n,m}} & + & \boxed{\substack{1,2;1,m\\n,2;n,m}} & + \cdots + & \boxed{\substack{1,m\\n,m}} \\
 & \vdots & \vdots & \vdots & & \vdots & \vdots & \vdots \\
+ & \boxed{\substack{n-1,2\\n,2}} & + \cdots + & \boxed{\substack{n-1,2;n-1,m\\n,2;n,m}} & + & \boxed{\substack{n-1,2;n-1,m\\n,2;n,m}} & + \cdots + & \boxed{\substack{n-1,m\\n,m}} \\
+ & \boxed{n,2} & + \cdots + & \boxed{n,2;n,m} & + & \boxed{n,2;n,m} & + \cdots + & \boxed{n,m}
\end{array} \tag{6}
$$

And finally, taking $a_{i,1} = 0$ and $a_{1,j}$ for all i, j, we have

$$\sum_{i=2}^{n}\sum_{j=2}^{m} a_{i,j} = \begin{array}{l}
\left.\underline{2,2}\right| + \cdots + \left.\underline{2,2;2,m}\right| + \left|\underline{2,2;2,m}\right. + \cdots + \left|\underline{2,m}\right. \\
+ \left.\underline{\begin{matrix}2,2\\3,2\end{matrix}}\right| + \cdots + \left.\underline{\begin{matrix}2,2;2,m\\3,2;3,m\end{matrix}}\right| + \left|\underline{\begin{matrix}2,2;2,m\\3,2;3,m\end{matrix}}\right. + \cdots + \left|\underline{\begin{matrix}2,m\\3,m\end{matrix}}\right. \\
\quad\vdots \qquad \vdots \qquad \vdots \qquad \vdots \qquad \vdots \qquad \vdots \\
+ \left.\underline{\begin{matrix}2,2\\n,2\end{matrix}}\right| + \cdots + \left.\underline{\begin{matrix}2,2;2,m\\n,2;n,m\end{matrix}}\right| + \left|\underline{\begin{matrix}2,2;2,m\\n,2;n,m\end{matrix}}\right. + \cdots + \left|\underline{\begin{matrix}2,m\\n,m\end{matrix}}\right. \\
+ \left.\overline{\begin{matrix}2,2\\n,2\end{matrix}}\right| + \cdots + \left.\overline{\begin{matrix}2,2;2,m\\n,2;n,m\end{matrix}}\right| + \left|\overline{\begin{matrix}2,2;2,m\\n,2;n,m\end{matrix}}\right. + \cdots + \left|\overline{\begin{matrix}2,m\\n,m\end{matrix}}\right. \\
\quad\vdots \qquad \vdots \qquad \vdots \qquad \vdots \qquad \vdots \qquad \vdots \\
+ \left.\overline{\begin{matrix}n-1,2\\n,2\end{matrix}}\right| + \cdots + \left.\overline{\begin{matrix}n-1,2;n-1,m\\n,2;n,m\end{matrix}}\right| + \left|\overline{\begin{matrix}n-1,2;n-1,m\\n,2;n,m\end{matrix}}\right. + \cdots + \left|\overline{\begin{matrix}n-1,m\\n,m\end{matrix}}\right. \\
+ \left.\overline{n,2}\right| + \cdots + \left.\overline{n,2;n,m}\right| + \left|\overline{n,2;n,m}\right. + \cdots + \left|\overline{n,m}\right.
\end{array} \tag{7}$$

Now, taking (4)-(5)-(6)+(7), we get the following necessary **and sufficient** condition (I was unable to fit it into the page in a way similar to the previous equations...)

$$\begin{aligned}
a_{1,1} = {} & \boxed{\begin{matrix}1,1;1,m\\n,1;n,m\end{matrix}} + \sum_{i=1}^{n-1}\sum_{j=1}^{m-1}\left(\left.\underline{\begin{matrix}1,1;1,j\\i,1;i,j\end{matrix}}\right| + \left.\underline{\begin{matrix}2,2;2,j+1\\i+1,2;i+1,j+1\end{matrix}}\right| - \left.\underline{\begin{matrix}1,2;1,j+1\\i,2;i,j+1\end{matrix}}\right| - \left.\underline{\begin{matrix}2,1;2,j\\i+1,1;i+1,j\end{matrix}}\right| \right) \\
& + \sum_{i=1}^{n-1}\left(\left|\underline{\begin{matrix}1,1;1,m\\i,1;i,m\end{matrix}}\right| - \left|\underline{\begin{matrix}2,1;2,m\\i+1,1;i+1,m\end{matrix}}\right| \right) + \sum_{j=1}^{m-1}\left(\left.\overline{\underline{\begin{matrix}1,1;1,j\\n,1;n,j\end{matrix}}}\right| - \left.\overline{\underline{\begin{matrix}1,2;1,j+1\\n,2;n,j+1\end{matrix}}}\right| \right)
\end{aligned}$$

Unfortunately, the resulting systems are beyond the computational power of my PC (and probably of any PC), for most sizes and numbers of states. The degrees of freedom for different sizes, for small Q, are shown in table 1.4.

For $n = m = 2$, the solutions were already seen in the previous sections. For $Q = 2$, $n = 2$, $m = 3$, 10 new solutions appeared.

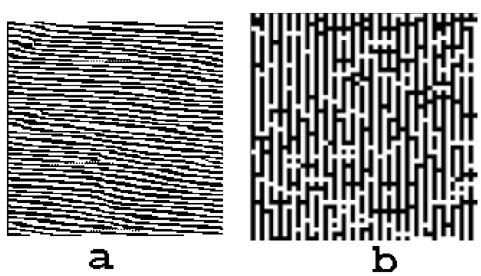

Fig. 8: Rules number 10 and 1, with neighbourhood size 6

Description of the rules

At least at first look, the only rule that seems to show something new is the self-conjugated rule 10, which leads to a stripe wrapped around the thorus, with different curvatures, according to the initial

Q	n	m	Degrees of freedom
2	1	2	1
	1	3	3
	1	4	7
	1	5	15
	1	6	31
	1	7	63
	2	2	5
	2	3	19
	2	4	71
	2	5	271
	3	3	111
3	1	2	2
	1	3	8
	1	4	26
	1	5	80
	2	2	14
	2	3	98
4	2	2	27
5	2	2	44

Tab. 10: Degrees of freedom for different sizes and Q.

1	0000000000000000010001001111111111001111110011110100010011111111
2	0000000000000000001000101111111111110011111100110010001011111111
3	0000000000000000011001101111111111100001110000110110011011111111
4	0000000000000000000100011111111111111100111111000001000111111111
5	0000000000000000100110011111111100111100001111001001100111111111
6	0000000000000000110111011111111100001100000011001101110111111111
7	0000000000100010110100011110011000011000010111011011101111111111
8	0000000000000000101110111111111100110000001100001011101111111111
9	0000000000000000011101111111111111100000011000000011101111111111
10	0000000000010001011101001111110011000000011010001011101111111111

Tab. 11: New rules for neighbourhood 2×3, $Q = 2$

condition. The rest of the rules have rather trivial behaviours; the picture given by rule 1 is new too (with respect to all the rules previously found).