Limit Theorems for Individual-Based Models in Economics and Finance

Daniel Remenik
Cornell University

January 12, 2009

Abstract

There is a widespread recent interest in using ideas from statistical physics to model certain types of problems in economics and finance. The main idea is to derive the macroscopic behavior of the market from the random local interactions between agents. Our purpose is to present a general framework that encompasses a broad range of models, by proving a law of large numbers and a central limit theorem for certain interacting particle systems with very general state spaces. To do this we draw inspiration from some work done in mathematical ecology and mathematical physics. The first result is proved for the system seen as a measure-valued process, while to prove the second one we will need to introduce a chain of embeddings of some abstract Banach and Hilbert spaces of test functions and prove that the fluctuations converge to the solution of a certain generalized Gaussian stochastic differential equation taking values in the dual of one of these spaces.

1 Introduction

We consider interacting particle systems of the following form. There is a fixed number N of particles, each one having a type $w \in W$. The particles change their types via two mechanisms. The first one corresponds simply to transitions from one type to another at some given rate. The second one involves a direct interaction between particles: pairs of particles interact at a certain rate and acquire new types according to some given (random) rule. We will allow these rates to depend directly on the types of the particles involved and on the distribution of the whole population on the type space.

Our purpose is to prove limit theorems, as the number of particles N goes to infinity, for the empirical random measures ν_t^N associated to these systems. ν_t^N is defined as follows: if $\eta_t^N(i) \in W$ denotes the type of the i-th particle at time t, then

$$\nu_t^N = \frac{1}{N}\sum_{i=1}^{N} \delta_{\eta_t^N(i)},$$

MSC: 60K35, 60B12, 46N30, 62P50, 91B70
Keywords: Individual-based model, interacting particle system, law of large numbers, central limit theorem, fluctuations process, measure-valued process, finance, economics

where δ_w is the probability measure on W assigning mass 1 to w.

Our first result, Theorem 1, provides a law of large numbers for ν_t^N on a finite time interval $[0,T]$: the empirical measures converge in distribution to a deterministic continuous path ν_t in the space of probability measures on W, whose evolution is described by a certain system of integro-differential equations. Theorem 2 analyzes the fluctuations of the finite system ν_t^N around ν_t, and provides an appropriate central limit result: the fluctuations are of order $1/\sqrt{N}$, and the asymptotic behavior of the process $\sqrt{N}\big(\nu_t^N - \nu_t\big)$ has a Gaussian nature. This second result is, as could be expected, much more delicate than the first one.

In recent years there has been an increasing interest in the use of interacting particle systems to model phenomena outside their original application to statistical physics, with special attention given to models in ecology, economics, and finance. Our model is specially suited for the last two types of problems, in particular because we have assumed a constant number of particles, which may represent agents in the economy or financial market (ecological problems, on the other hand, usually require including birth and death of particles). Particle systems were first used in this context in Föllmer (1974), and they have been used recently by many authors to analyze a variety of problems in economics and finance. The techniques that have been used are diverse, including, for instance, ideas taken from the Ising model in Föllmer (1974), the voter model in Giesecke and Weber (2004), the contact process in Huck and Kosfeld (2007), the theory of large deviations in Dai Pra, Runggaldier, Sartori, and Tolotti (2007), and the theory of queuing networks in Davis and Esparragoza-Rodriguez (2007) and Bayraktar, Horst, and Sircar (2007).

Our original motivation for this work comes precisely from financial modeling. It is related to some problems studied by Darrell Duffie and coauthors (see Examples 2.1 and 3.3) in which they derive some models from the random local interactions between the financial agents involved, based on the ideas of Duffie and Sun (2007). Our initial goal was to provide a general framework in which this type of problems could be rigorously analyzed, and in particular prove a law of large numbers for them. In our general setting, W will be allowed to be any locally compact complete separable metric space. Considering type spaces of this generality is one of the main features of our model, and it allows us to provide a unified framework to deal with models of different nature (for instance, the model in Example 2.1 has a finite type space and the limit solves a finite system of ordinary differential equations, while in Example 3.3 the type space is $\mathbb{R}$ and the limit solves a system of uncountably many integro-differential equations).

To achieve this first goal, we based our model and techniques on ideas taken from the mathematical biology literature, and in particular on Fournier and Méléard (2004), where the authors study a model that describes a spatial ecological system where plants disperse seeds and die at rates that depend on the local population density, and obtain a deterministic limit similar to ours. We remark that, following their ideas, our results could be extended to systems with a non-constant population by adding assumptions which allow to control the growth of the population, but we have preferred to keep this part of the problem simple.

The central limit result arose as a natural extension of this original question, but, as we already mentioned, it is much more delicate. The extra technical difficulties are related with the fact that the fluctuations of the process are signed measures (as opposed to the process ν_t^N which takes values in a space of probability measures), and the space of signed measures is not well suited for the study of convergence in distribution. The natural topology to consider for this space in our setting, that of weak convergence, is in general not metrizable. One could try to regard this space as the Banach space dual of the space of continuous bounded functions on W and endow it with its operator norm, but this topology is too strong in general to obtain tightness for the fluctuations (observe that, in particular, the total mass of the fluctuations $\sqrt{N}\big(\nu_t^N - \nu_t\big)$ is not a priori bounded uniformly in N). To overcome this difficulty we will show convergence of the fluctuations as a process taking values in the dual of a suitable abstract Hilbert space of test functions. We will actually have to consider a sequence of embeddings of Banach and Hilbert spaces, which will help us in controlling the norm of the fluctuations. This approach is inspired by ideas introduced in Métivier (1987) to study weak convergence of some measure-valued processes using sequences of Sobolev embeddings. Our proof is based on Méléard (1998), where the author proves a similar central limit result for a system of interacting diffusions associated with Boltzmann equations.

The rest of the paper is organized as follows. Section 2 contains the description of the general model, Section 3 presents the law of large numbers for our system, and Section 4 presents the central limit theorem, together with the description of the extra assumptions and the functional analytical setting we will use to obtain it. All the proofs are contained in Section 5.

2 Description of the Model

2.1 Introductory example

To introduce the basic features of our model and fix some ideas, we begin by presenting one of the basic examples we have in mind.

Example 2.1. We consider the model for over-the-counter markets introduced in Duffie, Gârleanu, and Pedersen (2005). There is a "consol", which is an asset paying dividends at a constant rate of 1, and there are N investors that can hold up to one unit of the asset. The total number of units of the asset remains constant in time, and the asset can be traded when the investors contact each other and when they are contacted by marketmakers. Each investor is characterized by whether he or she owns the asset or not, and by an intrinsic type that is "high" or "low". Low-type investors have a holding cost when owning the asset, while high-type investors do not. These characteristics will be represented by the set of types $W = \{ho, hn, lo, ln\}$, where h and l designate the high- and low-type of an investor while o and n designate whether an investor owns or not the asset.

At some fixed rate λ_d, high-type investors change their type to low. This means that each investor runs a Poisson process with rate λ_d (independent from the others), and at each event of this process the investor changes his or her intrinsic type to low (nothing happens if the investor is already of low-type). Analogously, low-type investors change to high-type at some rate λ_u. The meetings between agents are defined as follows: each investor decides to look for another investor at rate β (understood as before, i.e., at the times of the events of a Poisson process with rate β), chooses the investor uniformly among the set of N investors, and tries to trade. Additionally, each investor contacts a marketmaker at rate ρ. The marketmakers pair potential buyers and sellers, and the model assumes that this pairing happens instantly. At equilibrium, the rate at which investors trade through marketmakers is ρ times the minimum between the fraction of investors willing to buy and the fraction of investors willing to sell (see Duffie *et al.* (2005) for more details). In this model, the only encounters leading to a trade are those between *hn*- and *lo*-agents, since high-type investors not owning the asset are the only ones willing to buy, while low-type investors owning the asset are the only ones willing to sell.

Theorem 1 will imply the following for this model: as N goes to infinity, the (random) evolution of the fraction of agents of each type converges to a deterministic limit which is the unique solution of the following system of ordinary differential equations:

$$
\begin{aligned}
\dot{u}_{ho}(t) &= 2\beta u_{hn}(t)u_{lo}(t) + \rho \min\{u_{hn}(t), u_{lo}(t)\} + \lambda_u u_{lo}(t) - \lambda_d u_{ho}(t),\\
\dot{u}_{hn}(t) &= -2\beta u_{hn}(t)u_{lo}(t) - \rho \min\{u_{hn}(t), u_{lo}(t)\} + \lambda_u u_{ln}(t) - \lambda_d u_{hn}(t),\\
\dot{u}_{lo}(t) &= -2\beta u_{hn}(t)u_{lo}(t) - \rho \min\{u_{hn}(t), u_{lo}(t)\} - \lambda_u u_{lo}(t) + \lambda_d u_{ho}(t),\\
\dot{u}_{ln}(t) &= 2\beta u_{hn}(t)u_{lo}(t) + \rho \min\{u_{hn}(t), u_{lo}(t)\} - \lambda_u u_{ln}(t) + \lambda_d u_{hn}(t).
\end{aligned}
\tag{2.1}
$$

Here $u_w(t)$ denotes the fraction of type-w investors at time t. This deterministic limit corresponds to the one proposed in Duffie *et al.* (2005) for this model (see the referred paper for the interpretation of this equations and more on this model).

2.2 Description of the General Model

We will denote by $I_N = \{1, \dots, N\}$ the set of particles in the system. In line with our original financial motivation, we will refer to these particles as the "agents" in the system (like the investors of the aforementioned example). The possible types for the agents will be represented by a locally compact Polish (i.e., separable, complete, metrizable) space W. Given a metric space E, $\mathcal{P}(E)$ will denote the collection of probability measures on E, which will be endowed with the topology of weak convergence. When $E = W$, we will simply write $\mathcal{P} = \mathcal{P}(W)$. We will denote by $\mathcal{P}_a$ the subset of $\mathcal{P}$ consisting of purely atomic measures.

The Markov process ν_t^N we are interested in takes values in $\mathcal{P}_a$ and describes the evolution of the distribution of the agents over the set of types. We recall that it is defined as

$$\nu_t^N = \frac{1}{N}\sum_{i=1}^{N} \delta_{\eta_t^N(i)},$$

where δ_w is the probability measure on W assigning mass 1 to $w \in W$ and $\eta_t^N(i)$ corresponds to the type of the agent i at time t. In other words, the vector $\eta_t^N \in W^{I_N}$ gives the configuration of the set of agents at time t, while for any Borel subset A of W, $\nu_t^N(A)$ is the fraction of agents whose type is in A at time t.

The dynamics of the process is defined by the following rates:

- Each agent decides to change its type at a certain rate $\gamma(w, \nu_t^N)$ that depends on its current type w and the current distribution ν_t^N. The new type is chosen according to a probability measure $a(w, \nu_t^N, dw')$ on W.
- Each agent contacts each other agent at a certain rate that depends on their current types w_1 and w_2 and the current distribution ν_t^N: the total rate at which a given type-w_1 agent contacts type-w_2 agents is given by $N\lambda(w_1, w_2, \nu_t^N)\nu_t^N(\{w_2\})$. After a pair

of agents meet, they choose together a new pair of types according to a probability measure $b(w_1, w_2, \nu_t^N, dw_1' \otimes dw_2')$ (not necessarily symmetric in w_1, w_2) on $W \times W$.

For a fixed $\mu \in \mathcal{P}_a$, $a(w, \mu, dw')$ and $b(w_1, w_2, \mu, dw_1' \otimes dw_2')$ can be interpreted, respectively, as the transition kernels of Markov chains in W and $W \times W$.

Let $\mathcal{B}(W)$ be the collection of bounded measurable functions on W and $\mathcal{C}_b(W)$ be the collection of bounded continuous functions on W. For $\nu \in \mathcal{P}$ and $\varphi \in \mathcal{B}(W)$ (or, more generally, any measurable function φ) we write

$$\langle \nu, \varphi \rangle = \int_W \varphi \, d\nu.$$

Observe that

$$\langle \nu_t^N, \varphi \rangle = \frac{1}{N} \sum_{i=1}^N \varphi(\eta_t^N(i)).$$

We make the following assumption:

Assumption A.

(A1) The rate functions $\gamma(w, \nu)$ and $\lambda(w, w', \nu)$ are defined for all $\nu \in \mathcal{P}$. They are non-negative, measurable in w and w', bounded respectively by constants $\overline{\gamma}$ and $\overline{\lambda}$, and continuous in ν.

(A2) $a(w, \nu, \cdot)$ and $b(w, w', \nu, \cdot)$ are measurable in w and w'.

(A3) The mappings

$$\nu \longmapsto \int_W \gamma(w, \nu) \, a(w, \nu, \cdot) \, \nu(dw) \qquad \text{and}$$
$$\nu \longmapsto \int_W \int_W \lambda(w_1, w_2, \nu) \, b(w_1, w_2, \nu, \cdot) \, \nu(dw_2) \, \nu(dw_1),$$

which assign to each $\nu \in \mathcal{P}_a$ a finite measure on W and $W \times W$, respectively, are continuous with respect to the topology of weak convergence and Lipschitz with respect to the total variation norm: there are constants $C_a, C_b > 0$ such that

$$\left\| \int_W \gamma(w, \nu_1) a(w, \nu_1, \cdot) \, \nu_1(dw) - \int_W \gamma(w, \nu_2) a(w, \nu_2, \cdot) \, \nu_2(dw) \right\|_{\mathrm{TV}} \le C_a \|\nu_1 - \nu_2\|_{\mathrm{TV}}$$

and

$$\Bigg\| \int_W \int_W \lambda(w_1, w_2, \nu_1) b(w_1, w_2, \nu_1, \cdot) \, \nu_1(dw_2) \, \nu_1(dw_1)$$
$$- \int_W \int_W \lambda(w_1, w_2, \nu_2) b(w_1, w_2, \nu_2, \cdot) \, \nu_2(dw_2) \, \nu_2(dw_1) \Bigg\|_{\mathrm{TV}} \le C_b \|\nu_1 - \nu_2\|_{\mathrm{TV}}.$$

We recall that the total variation norm of a signed measure μ is defined by

$$\|\mu\|_{\mathrm{TV}} = \sup_{\varphi : \, \|\varphi\|_\infty \le 1} |\langle \mu, \varphi \rangle| \, .$$

(A3) is satisfied, in particular, whenever the rates do not depend on ν.

3 Law of large numbers for ν_t^N

Our first result shows that the process ν_t^N converges in distribution, as the number of agents N goes to infinity, to a deterministic limit that is characterized by a measure-valued system of differential equations (written in its weak form).

Given a metric space S, we will denote by $D([0,T],S)$ the space of càdlàg functions $\nu : [0,T] \longrightarrow S$, and we endow these spaces with the Skorohod topology (see Ethier and Kurtz (1986) or Billingsley (1999) for a reference on this topology and weak convergence in general). Observe that our processes ν_t^N have paths on $D([0,T],\mathcal{P})$ (recall that we are endowing $\mathcal{P}$ with the topology of weak convergence, which is metrizable). We will also denote by $C([0,T],S)$ the space of continuous functions $\nu\colon [0,T] \longrightarrow S$.

Theorem 1. *Suppose that Assumption A holds. For any given $T > 0$, consider the sequence of $\mathcal{P}$-valued processes ν_t^N on $[0,T]$, and assume that the sequence of initial distributions ν_0^N converges in distribution to some fixed $\nu_0 \in \mathcal{P}$. Then the sequence ν_t^N converges in distribution in $D([0,T],\mathcal{P})$ to a deterministic ν_t in $C([0,T],\mathcal{P})$, which is the unique solution of the following system of integro-differential equations: for every $\varphi \in \mathcal{B}(W)$ and $t \in [0,T]$,*

$$\begin{aligned}
\langle \nu_t, \varphi\rangle = \langle \nu_0, \varphi\rangle &+ \int_0^t \int_W \gamma(w,\nu_s) \int_W (\varphi(w') - \varphi(w))\, a(w,\nu_s,dw')\,\nu_s(dw)\,ds \\
&+ \int_0^t \int_W \int_W \lambda(w_1,w_2,\nu_s) \int_{W\times W} (\varphi(w_1') + \varphi(w_2') - \varphi(w_1) - \varphi(w_2)) \\
&\qquad \cdot b(w_1,w_2,\nu_s,dw_1'\otimes dw_2')\,\nu_s(dw_2)\,\nu_s(dw_1)\,ds.
\end{aligned} \tag{S1}$$

Observe that, in particular, (S1) implies that for every Borel set $A \subseteq W$ and almost every $t \in [0,T]$,

$$\begin{aligned}
\frac{d\nu_t(A)}{dt} &= -\int_A \left(\gamma(w,\nu_t) + \int_W \big(\lambda(w,w',\nu_t) + \lambda(w',w,\nu_t)\big)\nu_t(dw')\right)\nu_t(dw) \\
&\quad + \int_W \gamma(w,\nu_t) a(w,\nu_t,A)\,\nu_t(dw) \\
&\quad + \int_W\int_W \lambda(w,w',\nu_t)\Big[b(w,w',\nu_t,A\times W) + b(w,w',\nu_t,W\times A)\Big]\,\nu_t(dw')\,\nu_t(dw).
\end{aligned} \tag{S1$'$}$$

Furthermore, standard measure theory arguments allow to show that the system (S1′) actually characterizes the solution of (S1) (by approximating the test functions φ in (S1) by simple functions).

(S1′) has an intuitive interpretation: the first term on the right side is the total rate at which agents leave the set of types A, the second term is the rate at which agents decide to change their types to a type in A, and the third term is the rate at which agents acquire types in A due to interactions between them.

The following corollary of the previous result is useful when writing and analyzing the limiting equations (S1) or (S1′) (see, for instance, Example 3.3).

Corollary 3.1. *In the context of Theorem 1, assume that ν_0 is absolutely continuous with respect to some measure μ on W and that the measures*

$$\int_W \gamma(w,\nu_0)a(w,\nu_0,\cdot)\,\nu_0(dw) \quad \textit{and} \quad \int_W\int_W \lambda(w_1,w_2,\nu_0)b(w_1,w_2,\nu_0,\cdot)\,\nu_0(dw_1)\,\nu_0(dw_2)$$

are absolutely continuous with respect to μ and $\mu\otimes\mu$, respectively. Then the limit ν_t is absolutely continuous with respect to μ for all $t\in[0,T]$.

The following two examples show two different kinds of models: one with a finite type space and the other with $W=\mathbb{R}$. The first model is the one given in Example 2.1.

Example 3.2 (Continuation of Example 2.1)**.** To translate into our framework the model for over-the-counter markets of Duffie *et al.* (2005), we take $W=\{ho,hn,lo,ln\}$ and consider a set of parameters γ, a, λ, and b with all but λ being independent of ν_t^N. Let

$$\begin{aligned} \gamma(ho)&=\gamma(hn)=\lambda_d, & a(ho,\cdot)&=\delta_{lo}, & a(hn,\cdot)&=\delta_{ln},\\ \gamma(lo)&=\gamma(ln)=\lambda_u, & a(lo,\cdot)&=\delta_{ho}, & a(ln,\cdot)&=\delta_{hn}. \end{aligned}$$

Observe that with this definition, high-type investors become low-type at rate λ_d and low-type investors become high-type at rate λ_u, just as required. For the encounters between agents we take

$$\lambda(hn,lo,\nu)=\lambda(lo,hn,\nu)=\begin{cases}\beta+\frac{\rho}{2}\frac{\nu(\{hn\})\wedge\nu(\{lo\})}{\nu(\{hn\})\nu(\{lo\})} & \text{if } \nu(\{hn\})\nu(\{lo\})>0,\\ \beta & \text{if } \nu(\{hn\})\nu(\{lo\})=0,\end{cases}$$
$$b(hn,lo,\nu,\cdot)=\delta_{(ho,ln)},\qquad \text{and}\qquad b(lo,hn,\nu,\cdot)=\delta_{(ln,ho)}$$

(where $a\wedge b=\min\{a,b\}$), and for all other pairs $w_1,w_2\in W$, $\lambda(w_1,w_2,\nu)=0$ (recall that the only encounters leading to a trade are those between hn- and lo-agents and vice versa, in which case trade always occurs). The rates $\lambda(hn,lo,\nu)$ and $\lambda(lo,hn,\nu)$ have two terms: the rate β corresponding to the rate at which hn-agents contact lo-agents, plus a second rate reflecting trades carried out via a marketmaker. The form of this second rate assures that hn- and lo- agents meet through marketmakers at the right rate of $\rho\,\nu(\{hn\})\wedge\nu(\{lo\})$. It is not difficult to check that these parameters satisfy Assumption A, using the fact that $x\wedge y=(x+y-|x-y|)/2$ for $x,y\in\mathbb{R}$.

Now let $u_w(t)=\nu_t(\{w\})$, where ν_t is the limit of ν_t^N given by Theorem 1. We need to compute the right side of (S1′) with $A=\{w\}$ for each $w\in W$. Take, for example, $w=ho$. We get

$$\dot u_{ho}(t)=\lambda_u u_{lo}(t)-\lambda_d u_{ho}(t)+\beta u_{hn}(t)u_{lo}(t)+\frac{\rho}{2}u_{hn}(t)\wedge u_{lo}(t)+\beta u_{lo}(t)u_{hn}(t)+\frac{\rho}{2}u_{hn}(t)\wedge u_{lo}(t),$$

which corresponds exactly to the first equation in (2.1). The other three equations follow similarly.

Example 3.3. Our second example is based on the model for information percolation in large markets introduced in Duffie and Manso (2007). We will only describe the basic features of the model, for more details see the cited paper. There is a random variable X of concern to all agents which has two possible values, "high" or "low". Each agent holds some information about the outcome of X, and this information is summarized in a real number x which is a sufficient statistic for the posterior probability assigned by the agent (given his or her information) to the outcome of X being high. We take these statistics as the types of the agents (so $W = \mathbb{R}$). The model is set up so that these statistics satisfy the following: after a type-x_1 agent and a type-x_2 agent meet and share their information, $x_1 + x_2$ becomes a sufficient statistic for the posterior distribution of X assigned by both agents given now their shared information.

In this model the agents change types only after contacting other agents, so we take $\gamma \equiv 0$, and encounters between agents occur at a constant rate $\lambda > 0$. The transition kernel for the types of the agents after encounters is independent of ν_t^N and is given by

$$b(x_1, x_2, \cdot) = b(x_2, x_1, \cdot) = \delta_{(x_1+x_2, x_1+x_2)}$$

for every $x_1, x_2 \in \mathbb{R}$. This choice for the parameters trivially satisfies Assumption A.

To compute the limit of the process, let A be a Borel subset of $\mathbb{R}$. Then, since $\gamma \equiv 0$ and λ is constant, (S1′) gives

$$\begin{aligned}\dot{\nu}_t(A) &= -2\lambda\nu_t(A) + \lambda \int_{\mathbb{R}^2} \left(\delta_{(x+y,x+y)}(\mathbb{R}\times A) + \delta_{(x+y,x+y)}(A\times\mathbb{R})\right) \nu_t(dy)\, \nu_t(dx) \\ &= -2\lambda\nu_t(A) + 2\lambda \int_{\mathbb{R}^2} \delta_{x+y}(A)\, \nu_t(dy)\, \nu_t(dx) = -2\lambda\nu_t(A) + 2\lambda \int_{-\infty}^{\infty} \nu_t(A-x)\, \nu_t(dx),\end{aligned}$$

where $A - x = \{y \in \mathbb{R} \colon\ y + x \in A\}$. Therefore,

$$\dot{\nu}_t(A) = -2\lambda\nu_t(A) + 2\lambda(\nu_t * \nu_t)(A). \tag{3.1}$$

Using Corollary 3.1 we can write the last equation in a nicer form: if we assume that the initial condition ν_0 is absolutely continuous with respect to the Lebesgue measure, then the measures ν_t have a density g_t with respect to the Lebesgue measure, and we obtain

$$\dot{g}_t(x) = -2\lambda g_t(x) + 2\lambda \int_{-\infty}^{\infty} g_t(z-x) g_t(z)\, dz = -2\lambda g_t(x) + 2\lambda(g_t * g_t)(x).$$

This is the system of integro-differential equations proposed in Duffie and Manso (2007) for this model (except for the factor of 2, which is omitted in that paper).

4 Central limit theorem for ν_t^N

Theorem 1 gives the law of large numbers for ν_t^N, in the sense that it obtains a deterministic limit for the process as the size of the market goes to infinity. We will see now that, under some additional hypotheses, we can also obtain a central limit result for our process: the fluctuations of ν_t^N around the limit ν_t are of order $1/\sqrt{N}$, and they have, asymptotically, a Gaussian nature. As we mentioned in the Introduction, this result is much more delicate than Theorem 1, and we will need to work hard to find the right setting for it.

The *fluctuations* process is defined as follows:

$$\sigma_t^N = \sqrt{N}\big(\nu_t^N - \nu_t\big).$$

σ_t^N is a sequence of finite signed measures, and our goal is to prove that it converges to the solution of a system of stochastic differential equations driven by a Gaussian process. As we explained in the Introduction, regarding the fluctuations process as taking values in the space of signed measures, and endowing this space with the topology of weak convergence (which corresponds to seeing the process as taking values in the Banach space dual of $\mathcal{C}_b(W)$ topologized with the weak* convergence) is not the right approach for this problem. The idea will be to replace the test function space $\mathcal{C}_b(W)$ by an appropriate Hilbert space $\mathcal{H}_1$ and regard σ_t^N as a linear functional acting on this space via the mapping $\varphi \longmapsto \langle \sigma_t^N, \varphi \rangle$. In other words, we will regard σ_t^N as a process taking values in the dual $\mathcal{H}_1{}'$ of a Hilbert space $\mathcal{H}_1$.

The space $\mathcal{H}_1$ that we choose will depend on the type space W. Actually, whenever W is not finite we will not need a single space, but a chain of seven spaces embedded in a certain structure. Our goal is to handle (at least) the following four possibilities for W: a finite set, $\mathbb{Z}^d$, a "sufficiently smooth" compact subset of $\mathbb{R}^d$, and all of $\mathbb{R}^d$. We wish to handle these cases under a unified framework, and this will require us to abstract the necessary assumptions on our seven spaces and the parameters of the model. We will do this in Sections 4.1 and 4.2, and then in Section 4.3 we will explain how to apply this abstract setting to the four type spaces W that we just mentioned.

4.1 General setting

During this and the next subsection we will assume as given the collection of spaces in which our problem will be embedded, and then we will make some assumptions on the parameters of our process that will assure that they are compatible with the structure of these spaces. The idea of this part is that we will try to impose as little as possible on these spaces, leaving their definition to be specified for the different cases of type space W.

The elements we will use are the following:

- Four separable Hilbert spaces of measurable functions on W, $\mathcal{H}_1$, $\mathcal{H}_2$, $\mathcal{H}_3$, and $\mathcal{H}_4$.
- Three Banach spaces of continuous functions on W, $\mathcal{C}_0, \mathcal{C}_2$, and $\mathcal{C}_3$.
- Five continuous functions $\rho_0, \rho_1, \rho_2, \rho_3, \rho_4 : W \longrightarrow [1, \infty)$ such that $\rho_i \leq \rho_{i+1}$ for $i = 0, 1, 2, 3$, $\rho_i \in \mathcal{C}_i$ for $i = 0, 2, 3$, and for all $w \in W$, $\rho_1^p(w) \leq C\rho_4(w)$ for some $C > 0$ and $p > 1$ (this last requirement is very mild, as we will see in the examples below, but will be necessary in the proof of Theorem 2).

The seven spaces and the five functions introduced above must be related in a specific way. First, we assume that the following sequence of continuous embeddings holds:

(B1) $$\mathcal{C}_0 \hookrightarrow \mathcal{H}_1 \underset{c}{\hookrightarrow} \mathcal{H}_2 \hookrightarrow \mathcal{C}_2 \hookrightarrow \mathcal{H}_3 \hookrightarrow \mathcal{C}_3 \hookrightarrow \mathcal{H}_4,$$

where the c under the second arrow means that the embedding is compact. We recall that a continuous embedding $E_1 \hookrightarrow E_2$ between two normed spaces E_1, E_2 implies, in particular, that $\|\cdot\|_{E_2} \leq C\|\cdot\|_{E_1}$ for some $C > 0$, while saying that the embedding is compact means that every bounded set in E_1 is compact in E_2.

Second, we assume that for $i = 1, 2, 3, 4$, if $\varphi \in \mathcal{H}_i$ then

(B2) $$|\varphi(w)| \leq C \, \|\varphi\|_{\mathcal{H}_i} \, \rho_i(w)$$

for all $w \in W$, for some $C > 0$ which does not depend on φ. The same holds for the spaces $\mathcal{C}_i$: for $i = 0, 2, 3$ and $\varphi \in \mathcal{C}_i$,

(B3) $$|\varphi(w)| \leq C \, \|\varphi\|_{\mathcal{C}_i} \, \rho_i(w).$$

The functions ρ_i will typically appear as weighting functions in the definition of the norms of the spaces $\mathcal{H}_i$ and $\mathcal{C}_i$. They will dictate the maximum growth rate allowed for functions in these spaces.

We will denote by $\mathcal{H}_i{}'$ and $\mathcal{C}_i{}'$ the topological duals of the spaces $\mathcal{H}_i$ and $\mathcal{C}_i$, respectively, endowed with their operator norms (in particular, the spaces $\mathcal{H}_i{}'$ and $\mathcal{C}_i{}'$ are Hilbert and Banach spaces themselves). Observe that (B1) implies the following dual continuous embeddings:

(B1′) $$\mathcal{H}_4{}' \hookrightarrow \mathcal{C}_3{}' \hookrightarrow \mathcal{H}_3{}' \hookrightarrow \mathcal{C}_2{}' \hookrightarrow \mathcal{H}_2{}' \underset{\mathrm{c}}{\hookrightarrow} \mathcal{H}_1{}' \hookrightarrow \mathcal{C}_0{}'.$$

Before continuing, let us describe briefly the main ideas behind the proof of our central limit theorem, which will help explain why this is a good setting for proving convergence of the fluctuations process. What we want to prove is that σ^N_t converges in distribution, as a process taking values in $\mathcal{H}_1{}'$, to the solution σ_t of a certain stochastic differential equation (see (S2) below). The approach we will take to prove this (the proof of Theorem 1 follows an analogous line) is standard: we first prove that the sequence σ^N_t is tight, then we show that any limit point of this sequence satisfies the desired stochastic differential equation, and finally we prove that this equation has a unique solution (in distribution). To achieve this we will follow the line of proof of Méléard (1998). Our sequence of embeddings (B1′) corresponds there to a sequence of embeddings of weighted Sobolev spaces (see (3.11) in the cited paper); in particular, we will use a very similar sequence of spaces to deal with the case $W = \mathbb{R}^d$ in Section 4.3.4. One important difficulty with this approach is the following: the operator J_s associated with the drift term of our limiting equation (see (4.1)), as well as the corresponding operators J^N_s for σ^N_t (see (5.9)), cannot in general be taken to be bounded as operators acting on any of the spaces $\mathcal{H}_i$. This forces us to introduce the spaces $\mathcal{C}_i$, on which (B3) plus some assumptions on the rates of the process will assure that J_s and J^N_s are bounded.

The scheme of proof will be roughly as follows. We will consider the semimartingale decomposition of the real-valued process $\langle \sigma^N_t, \varphi \rangle$, for $\varphi \in \mathcal{H}_4$, and then show that the martingale part defines a martingale in $\mathcal{H}_4{}'$. This, together with a moment estimate on the norm of the martingale part in $\mathcal{H}_4{}'$ and the boundedness of the operators J^N_s in $\mathcal{C}_3{}'$, will allow us to deduce that σ^N_t can be seen as a semimartingale in $\mathcal{H}_3{}'$, and moreover give its semimartingale decomposition. Next, we will give a uniform estimate (in N) of the norm of σ^N_t in $\mathcal{C}_2{}'$. This implies the same type of estimate in $\mathcal{H}_2{}'$, and this will allow us to obtain the tightness of σ^N_t in $\mathcal{H}_1{}'$. The fact that the embedding $\mathcal{H}_2{}' \hookrightarrow \mathcal{H}_1{}'$ is compact is crucial in this step. Then we will show that all limit points of σ^N_t have continuous paths in $\mathcal{H}_1{}'$ and they all satisfy the desired stochastic differential equation (S2). Unfortunately, it will not be possible to achieve this last part in $\mathcal{H}_1{}'$, due to the unboundedness of J_s in this space. Consequently, we are forced to embed the equation in the (bigger) space $\mathcal{C}_0{}'$. The boundedness of J_s in $\mathcal{C}_0{}'$ will also allow us to obtain uniqueness for the solutions of this equation in this space, thus finishing the proof.

Our last assumption (D below) will assure that our abstract setting is compatible with the rates defining our process. Before that, we need to replace Assumptions (A1) and (A2) by stronger versions:

Assumption C.

(C1) There is a family of finite measures $\big\{\Gamma(w,z,\cdot)\big\}_{w,z\in W}$ on W, whose total masses are bounded by $\overline{\gamma}$, such that for every $w \in W$ and every $\nu \in \mathcal{P}$ we have

$$\gamma(w,\nu)a(w,\nu,dw') = \int_W \Gamma(w,z,dw')\,\nu(dz).$$

$\Gamma(w,z,\cdot)$ is measurable in w and continuous in z.

(C2) There is a family of measures $\big\{\Lambda(w_1,w_2,z,\cdot)\big\}_{w_1,w_2,z\in W}$ on $W\times W$, whose total masses are bounded by $\overline{\lambda}$, such that for every $w_1,w_2 \in W$ and every $\nu \in \mathcal{P}$ we have

$$\lambda(w_1,w_2,\nu)b(w_1,w_2,\nu,dw_1'\otimes dw_2') = \int_{W\times W} \Lambda(w_1,w_2,z,dw_1'\otimes dw_2')\,\nu(dz).$$

$\Lambda(w_1,w_2,z,\cdot)$ is measurable in w_1 and w_2 and continuous in z.

The intuition behind this assumption is the following: the total rate at which a type-w agent becomes a type-w' agent is computed by averaging the effect that each agent in the market has on this rate for the given agent. Observe that, under this assumption, (A3) holds.

Remark 4.1. Assumption C has the effect of linearizing the jump rates in ν. This turns out to be very convenient, because it will allow us to express the drift term of the stochastic differential equation describing the limiting fluctuations σ_t ((S2) below) as $J_t\sigma_t$ for some $J_t \in \mathcal{C}_0{}'$ (see (4.1) and (5.9)). A more general approach would be to assume that the jump kernels, seen as operators acting on $\mathcal{C}_0{}'$, are Fréchet differentiable. In that case we would need to change the form of the drift operator J_t in the limiting equation and of Assumption D below, but the proof of Theorem 2 would still work, without any major modifications. To avoid extra complications, and since all the examples we have in mind satisfy Assumption C, we will restrict ourselves to this simpler case.

We introduce the following notation: given a measurable function φ on W, let

$$\Gamma\varphi(w;z) = \int_W (\varphi(w') - \varphi(w))\,\Gamma(w,z,dw') \qquad \text{and}$$

$$\Lambda\varphi(w_1,w_2;z) = \int_{W\times W} (\varphi(w_1') + \varphi(w_2') - \varphi(w_1) - \varphi(w_2))\,\Lambda(w_1,w_2,z,dw_1'\otimes dw_2').$$

These quantities can be thought of as the jump kernels for the process associated with the effect of a type-z agent on the transition rates. Averaging these rates with respect to $\nu_t^N(dz)$ gives the total jump kernel for the process.

Assumption D.

(D1) There is a $C > 0$ such that for all $w, z \in W$ and $i = 0, 1, 2, 3, 4$,

$$\int_W \rho_i^2(w')\,\Gamma(w, z, dw') < C\left(\rho_i^2(w) + \rho_i^2(z)\right).$$

(D2) There is a $C > 0$ such that for all $w_1, w_2, z \in W$ and $i = 0, 1, 2, 3, 4$,

$$\int_{W\times W} \left(\rho_i^2(w_1') + \rho_i^2(w_2')\right)\,\Lambda(w_1, w_2, z, dw_1' \otimes dw_2') < C\left(\rho_i^2(w_1) + \rho_i^2(w_2) + \rho_i^2(z)\right).$$

(D3) Let $\mu_1, \mu_2 \in \mathcal{P}$ be such that $\left\langle \mu_i, \rho_4^2 \right\rangle < \infty$ and define the following operator acting on measurable functions φ on W:

$$\begin{aligned} J_{\mu_1,\mu_2}\varphi(z) = &\int_W \Gamma\varphi(w; z)\,\mu_1(dw) + \int_W \Gamma\varphi(z; x)\,\mu_2(dx) \\ &+ \int_W\int_W \Lambda\varphi(w_1, w_2; z)\,\mu_1(dw_2)\,\mu_1(dw_1) + \int_W\int_W \Lambda\varphi(w, z; x)\,\mu_1(dw)\,\mu_2(dx) \\ &+ \int_W\int_W \Lambda\varphi(z, w; x)\,\mu_2(dw)\,\mu_2(dx). \end{aligned}$$

Then:

(i) J_{μ_1,μ_2} is a bounded operator on $\mathcal{C}_i$, for $i = 0, 2, 3$. Moreover, its norm can be bounded uniformly in μ_1, μ_2.

(ii) There is a $C > 0$ such that given any $\varphi \in \mathcal{C}_0$ and any $\mu_1, \mu_2, \mu_3, \mu_4 \in \mathcal{P}$ satisfying $\left\langle \mu_i, \rho_4^2 \right\rangle < \infty$,

$$\left\|\left(J_{\mu_1,\mu_2} - J_{\mu_3,\mu_4}\right)\varphi\right\|_{\mathcal{C}_0} \le C\,\|\varphi\|_{\mathcal{C}_0}\left(\|\mu_1 - \mu_3\|_{\mathcal{C}_2{}'} + \|\mu_2 - \mu_4\|_{\mathcal{C}_2{}'}\right).$$

(D1) and (D2) correspond to moment assumptions on the transition rates of the agents, and assure that the agents do not jump "too far". (D3.i) says two things: first, that the jump kernel defined by the rates preserves the structure of the spaces $\mathcal{C}_i$ and, second, that the resulting operator is bounded, which will imply the boundedness of the drift operators J_s and J_s^N mentioned above. (D3.ii) involves a sort of strengthening of the Lipschitz condition (A3) on the rates, and will be used to prove uniqueness for the limiting stochastic differential equation. Observe that by taking larger weighting functions ρ_i, which corresponds to taking smaller spaces of test functions $\mathcal{H}_i$, we add more moment assumptions on the rates of the process; on the other hand, asking for more structure on the spaces $\mathcal{H}_i$ and $\mathcal{C}_i$, such as differentiability in the Euclidean case, adds more requirements on the regularity of the rates.

4.2 Statement of the theorem

For $\xi \in \mathcal{H}_i{}'$ (respectively $\mathcal{C}_i{}'$) and $\varphi \in \mathcal{H}_i$ (respectively $\mathcal{C}_i$) we will write

$$\langle \xi, \varphi \rangle = \xi(\varphi).$$

Given $\varphi \in \mathcal{H}_1$ and $z \in W$ define

$$
\begin{aligned}
J_s\varphi(z) = \int_W \Gamma\varphi(w;z)\,\nu_s(dw) &+ \int_W \Gamma\varphi(z;x)\,\nu_s(dx) \\
&+ \int_W\int_W \Lambda\varphi(w_1,w_2;z)\,\nu_s(dw_2)\,\nu_s(dw_1) \\
&+ \int_W\int_W \big[\Lambda\varphi(z,w;x) + \Lambda\varphi(w,z;x)\big]\,\nu_s(dw)\,\nu_s(dx)
\end{aligned}
\tag{4.1}
$$

Observe that $J_s = J_{\nu_s,\nu_s}$. Therefore, under moment assumptions on ν_s, (D3.i) implies that J_s is a bounded operator on each of the spaces $\mathcal{C}_i$. Observe that if we integrate the first and third terms on the right side of (4.1) with respect to $\nu_s(dz)$, we obtain the integral term in (S1). In our central limit result, the variable z will be integrated against the limiting fluctuation process σ_t. The other two terms in (4.1) correspond to fluctuations arising from the dependence of the rates on its other arguments (the types of the agents involved).

The operator J_s (or, more properly, its adjoint J_s^*) will appear in the drift term of the stochastic differential equation describing the limiting fluctuations process, which will be expressed as a Bochner integral. We recall that these integrals are an extension of the Lebesgue integral to functions taking values on a Banach space, see Section V.5 in Yosida (1995) for details.

Theorem 2. *Assume that Assumptions C and D hold, that* (B1), (B2), *and* (B3) *hold, and that*

$$
\begin{aligned}
&\sqrt{N}(\nu_0^N - \nu_0) \Longrightarrow \sigma_0, \qquad \sup_{N\geq 0} \mathbb{E}\bigg(\Big\|\sqrt{N}\,(\nu_0^N - \nu_0)\Big\|^2_{\mathcal{C}_2{}'}\bigg) < \infty, \\
&\sup_{N\geq 0} \mathbb{E}\big(\langle \nu_0^N, \rho_4^2\rangle\big) < \infty, \qquad \textit{and} \qquad \mathbb{E}\big(\langle \nu_0, \rho_4^2\rangle\big) < \infty
\end{aligned}
\tag{4.2}
$$

hold, where the convergence in distribution above is in ${\mathcal{H}_1}'$. *Then the sequence of processes* σ_t^N *converges in distribution in* $D([0,T],{\mathcal{H}_1}')$ *to a process* $\sigma_t \in C([0,T],{\mathcal{H}_1}')$. *This process is the unique (in distribution) solution in* ${\mathcal{C}_0}'$ *of the following stochastic differential equation:*

$$
\sigma_t = \sigma_0 + \int_0^t J_s^*\sigma_s\,ds + Z_t, \tag{S2}
$$

where the above is a Bochner integral, J_s^* *is the adjoint of the operator* J_s *in* $\mathcal{C}_0$, *and* Z_t *is a centered* ${\mathcal{C}_0}'$*-valued Gaussian process with quadratic covariations specified by*

$$
\begin{aligned}
\big[Z_\cdot(\varphi_1), Z_\cdot(\varphi_2)\big]_t = &\int_0^t\int_W\int_W\int_W (\varphi_1(w') - \varphi_1(w))(\varphi_2(w') - \varphi_2(w))\,\Gamma(w,z,dw') \\
&\qquad\qquad\qquad\qquad \cdot \nu_s(dz)\,\nu_s(dw)\,ds \\
&+ \int_0^t\int_W\int_W\int_W\int_{W\times W} (\varphi_1(w_1') + \varphi_1(w_2') - \varphi_1(w_1) - \varphi_1(w_2)) \\
&\qquad \cdot (\varphi_2(w_1') + \varphi_2(w_2') - \varphi_2(w_1) - \varphi_2(w_2)) \\
&\qquad \cdot \Lambda(w_1,w_2,z,dw_1'\otimes dw_2')\,\nu_s(dz)\,\nu_s(dw_2)\,\nu_s(dw_1)\,ds
\end{aligned}
$$

for every $\varphi_1, \varphi_2 \in \mathcal{C}_0$.

We will denote by $C_s^{\varphi_1,\varphi_2}$ the sum of the two terms inside the time integrals above, so

$$\left[Z_\cdot(\varphi_1), Z_\cdot(\varphi_2)\right]_t = \int_0^t C_s^{\varphi_1,\varphi_2}\, ds.$$

Remark 4.2.

1. (S2) implies, in particular, that the solution σ_t satisfies

$$\langle \sigma_t, \varphi\rangle = \langle \sigma_0, \varphi\rangle + \int_0^t \langle \sigma_s, J_s\varphi\rangle\, ds + Z_t(\varphi) \tag{S2-w}$$

   simultaneously for every $\varphi \in \mathcal{C}_0$.

2. Observe that for any $\varphi_1, \ldots, \varphi_k \in \mathcal{C}_0$, the process $Z_t^{\varphi_1,\ldots,\varphi_k} = (Z_t(\varphi_1), \ldots, Z_t(\varphi_k))$ is a continuous $\mathbb{R}^k$-valued centered martingale with deterministic quadratic covariations, so it can be represented as

$$Z_t^{\varphi_1,\ldots,\varphi_k} \stackrel{d}{=} \int_0^t \left([C_s]^{\varphi_1,\ldots,\varphi_k}\right)^{1/2} dB_s,$$

   where $[C_t]^{\varphi_1,\ldots,\varphi_k}$ is the $k\times k$ matrix-valued process with entries given by $[C_t^{\varphi_1,\ldots,\varphi_k}]_{ij} = C_t^{\varphi_i,\varphi_j}$, $([C_t]^{\varphi_1,\ldots,\varphi_k})^{1/2}$ is the square root of this matrix, and B_t is a standard k-dimensional Brownian motion. Thus, writing $\langle \sigma_t; \varphi_1, \ldots, \varphi_k\rangle = \big(\langle \sigma_t, \varphi_1\rangle, \ldots, \langle \sigma_t, \varphi_k\rangle\big)$ we have

$$\langle \sigma_t; \varphi_1, \ldots, \varphi_k\rangle \stackrel{d}{=} \int_0^t \langle \sigma_t; J_s\varphi_1, \ldots, J_s\varphi_k\rangle\, ds + \int_0^t \left([C_s]^{\varphi_1,\ldots,\varphi_k}\right)^{1/2} dB_s. \tag{4.3}$$

3. The limiting fluctuations σ_t have zero mass in the following sense: whenever $\mathbf{1} \in \mathcal{C}_0$ and $\langle \sigma_0, \mathbf{1}\rangle = 0$, $\langle \sigma_t, \mathbf{1}\rangle = 0$ for all $t \in [0, T]$ almost surely. This follows from (4.3) simply by observing that, in this case, $J_s\mathbf{1}$ and $C_s^{\mathbf{1},\mathbf{1}}$ are both always zero.

Before presenting concrete examples where the setting and assumptions of this section hold, we present a general condition which allows to deduce that the assumptions (4.2) on the initial distributions ν_0^N, ν_0, and σ_0^N hold (namely, that ν_0^N is a product measure).

Theorem 3. *In the setting of Theorem 2, assume that ν_0^N is the product of N copies of a fixed probability measure $\nu_0 \in \mathcal{P}$ (i.e., ν_0^N is chosen by picking the initial type of each agent independently according to ν_0), and that $\mathbb{E}(\langle \nu_0, \rho_4^2\rangle) < \infty$. Then ν_0^N converges in distribution in $\mathcal{P}$ to ν_0, σ_0^N converges in distribution in ${\mathcal{H}_1}'$ to a centered Gaussian ${\mathcal{H}_1}'$-valued random variable σ_0, and all the assumptions in (4.2) are satisfied.*

4.3 Application to concrete type spaces

In this part we will present conditions under which the assumptions of Theorem 2 are satisfied in the four cases discussed at the beginning of this section.

4.3.1 Finite W

This is the easy case. The reason is that $\mathcal{C}_b(W)$ can be identified with $\mathbb{R}^{|W|}$, and thus σ_t^N can be regarded as an $\mathbb{R}^{|W|}$-valued process, so most of the technical issues disappear. In particular, Theorem 2 can be proved in this case by arguments very similar to those leading to Theorem 1.

In the abstract setting of Theorem 2, it is enough to choose $\rho_i \equiv 1$ and $\mathcal{H}_i = \mathcal{C}_i = \mathbb{R}^{|W|} \cong \ell^2(W)$ for the right indices $i \in \{0, 1, 2, 3, 4\}$ in each case. (B1) follows simply from the finite-dimensionality of $\mathbb{R}^{|W|}$ and the equivalence of all norms in finite dimensions and (B2), (B3), and Assumption D are satisfied trivially.

Theorem 2 takes a simpler form in this case. Write $W = \{w_1, \dots, w_k\}$,

$$\sigma_i^N(t) = \sigma_t^N(\{w_i\}), \qquad f_i(\sigma) = \sum_{j=1}^{k} J_s \mathbf{1}_{\{w_i\}}(w_j)\, \sigma_j, \quad \text{and} \qquad g_{ij}(t) = C_t^{\mathbf{1}_{\{w_i\}}, \mathbf{1}_{\{w_j\}}},$$

where σ above is in $\mathbb{R}^k$. Also write $F(\sigma) = \big(f_1(\sigma), \dots, f_k(\sigma)\big)$ and $G(t) = \big(g_{ij}(t)\big)_{i,j=1\dots,k}$. Observe that $G(t)$ is a positive semidefinite matrix for all $t \geq 0$.

Theorem 4a. *In the above context, assume that Assumption C holds and that*

$$\sqrt{N}\left(\nu_0^N - \nu_0\right) \Longrightarrow \sigma_0 \qquad \text{and} \qquad \sup_{N>0} \mathbb{E}\left(\left|\sqrt{N}\left(\nu_0^N - \nu_0\right)\right|^2\right) < \infty,$$

where the probability measures ν_t^N and ν_t are taken here as elements of $[0,1]^k$ and $\sigma_0 \in \mathbb{R}^k$. Then the sequence of processes $\sigma^N(t)$ converges in distribution in $D([0,T], \mathbb{R}^k)$ to the unique solution $\sigma(t)$ of the following system of stochastic differential equations:

$$d\sigma(t) = F(\sigma(t))\, dt + G^{1/2}(t)\, dB_t, \tag{S2-f}$$

where B_t is a standard k-dimensional Brownian motion.

Example 4.3. This example provides a very simple model of agents changing their opinions on some issue of common interest, with rates of change depending on the "popularity" of each alternative. These opinions will be represented by $W = \{-m, \dots, m\}$ (m can be thought of as being the strongest agreement with some idea, 0 as being neutral, and $-m$ as being the strongest disagreement with it). Alternatively, one could think of the model as describing the locations of the agents, who move according to the density of agents at each site.

The agents move in two ways. First, each agent feels attracted to other positions proportionally to the fraction of agents occupying them. Concretely, we assume that an agent at position i goes to position j at rate $\beta q_{i,j} \nu_t^N(\{j\})$, where $Q = (q_{i,j})_{i,j \in W}$ is the transition matrix of a Markov chain on W. One interpretation of these rates is that each agent decides to try to change its position at rate β, chooses a possible new position j according to Q, and then changes its position with probability $\nu_t(\{j\})$ and stays put with probability $1 - \nu_t(\{j\})$. Second, each agent leaves its position at a rate proportional to the fraction of agents at its own position. We assume then that, in addition to the previous rates, each agent at position i goes to position j at rate $\alpha p_{i,j} \nu_t^N(\{i\})$, where $P = (p_{i,j})_{i,j \in W}$ is defined analogously to Q. This can be thought of as the agent leaving its position i due to "overcrowding" at rate $\alpha\nu_t(\{i\})$ and choosing a new position according to P. We assume for simplicity that $p_{i,i} = q_{i,i} = 0$ for all $i \in W$.

We will set up the rates using the notation of Assumption C:

$$\Gamma(i,k,\{j\}) = \begin{cases} \alpha p_{i,j} & \text{if } k = i \\ \beta q_{i,j} & \text{if } k = j \\ 0 & \text{otherwise} \end{cases} \qquad \text{and} \qquad \Lambda \equiv 0.$$

Assume that ν_0^N converges in distribution to some $\nu_0 \in \mathcal{P}$, let ν_t be the limit given by Theorem 1 and write $u_t(i) = \nu_t(\{i\})$. It is easy to check that u_t satisfies

$$\frac{du_t(i)}{dt} = \alpha \sum_{j=-m}^{m} p_{j,i} u_t(j)^2 - \alpha u_t(i)^2 + \beta \sum_{j=-m}^{m} \big[q_{j,i} - q_{i,j}\big] u_t(i) u_t(j).$$

Now let σ_t be the limit in distribution of the fluctuations process $\sqrt{N}\big(u_t^N - u_t\big)$, and assume that the initial distributions ν_0^N and ν_0 satisfy the assumptions of Theorem 4a. It easy to check as before that

$$F_i(\sigma_t) = 2\alpha \sum_{j=-m}^{m} p_{j,i} u_t(j)\sigma_t(j) - 2\alpha u_t(i)\sigma_t(i) + \beta \sum_{j=-m}^{m} \big[q_{j,i} - q_{i,j}\big] \big(u_t(i)\sigma_t(j) + u_t(j)\sigma_t(i)\big).$$

Thus, after computing the quadratic covariations we obtain the following: if $\star$ denotes the coordinate-wise product in $\mathbb{R}^{|W|}$ (i.e., $u \star v(i) = u(i)v(i)$) then the limiting fluctuations process σ_t solves

$$\begin{aligned} d\sigma_t = 2\alpha P^{\mathrm{t}}(u_t \star \sigma_t)\, dt - 2\alpha u_t \star \sigma_t \, dt + \beta\Big(\big[Q^{\mathrm{t}} - Q\big]\sigma_t\Big) \star u_t \, dt \\ + \beta\Big(\big[Q^{\mathrm{t}} - Q\big] u_t\Big) \star \sigma_t \, dt + \sqrt{G(t)}\, dB_t, \end{aligned}$$

where B_t is a $(2m+1)$-dimensional standard Brownian motion and $G(t)$ is given by

$$G_{i,j}(t) = \begin{cases} \alpha \sum_{k \neq i} p_{k,i} u_t(k)^2 + \alpha u_t(i)^2 + \beta \sum_{k \neq i} \big(q_{k,i} + q_{i,k}\big) u_t(i) u_t(k) & \text{if } i = j \\ -\alpha\big(p_{j,i} u_t(j)^2 + p_{i,j} u_t(i)^2\big) - \beta\big(q_{j,i} + q_{i,j}\big) u_t(i) u_t(j) & \text{if } i \neq j. \end{cases}$$

4.3.2 $W = \mathbb{Z}^d$

In this case $\mathcal{C}_b(W)$ is no longer finite-dimensional and, moreover, the type space is not compact, so we will need to make use of the weighting functions ρ_i. We let $D = \lfloor d/2 \rfloor + 1$ and take

$$\rho_i(x) = \sqrt{1 + |x|^{2iD}}.$$

Clearly, we have in this case that $\rho_1^p \le C\rho_4$ for $C = p = 2$.

Consider the following spaces:

$$\mathcal{C}_0 = \ell^\infty(\mathbb{Z}^d) = \left\{\varphi\colon \mathbb{Z}^d \to \mathbb{R} \text{ such that } \|\varphi\|_\infty < \infty\right\},$$

$$\mathcal{C}_i = \ell^{\infty,iD}(\mathbb{Z}^d) = \left\{\varphi\colon \mathbb{Z}^d \to \mathbb{R} \text{ such that } \|\varphi\|_{\infty,iD} = \sup_{x\in\mathbb{Z}^d} \frac{|\varphi(x)|}{1+|x|^{iD}} < \infty\right\} \quad (i=2,3),$$

$$\mathcal{H}_i = \ell^{2,iD}(\mathbb{Z}^d) = \left\{\varphi\colon \mathbb{Z}^d \to \mathbb{R} \text{ such that } \|\varphi\|_{2,iD}^2 = \sum_{x\in\mathbb{Z}^d} \frac{|\varphi(x)|^2}{1+|x|^{2iD}} < \infty\right\} \quad (i=1,2,3,4),$$

endowed with the norms defined within these definitions (we observe that ρ_i does not appear explicitly in the definition of the spaces $\mathcal{C}_i$, but the definition does not change if we replace the weighting function $1+|x|^{iD}$ appearing there by ρ_i). These spaces are easily checked to be Banach (the $\mathcal{C}_i$) and Hilbert (the $\mathcal{H}_i$) as required. With these definitions we have the following continuous embeddings:

$$\begin{aligned} \ell^\infty(\mathbb{Z}^d) \hookrightarrow \ell^{2,D}(\mathbb{Z}^d) \underset{c}{\hookrightarrow} \ell^{2,2D}(\mathbb{Z}^d) \hookrightarrow \ell^{\infty,2D}(\mathbb{Z}^d) \hookrightarrow \ell^{2,3D}(\mathbb{Z}^d) \\ \hookrightarrow \ell^{\infty,3D}(\mathbb{Z}^d) \hookrightarrow \ell^{2,4D}(\mathbb{Z}^d) \end{aligned} \tag{4.4}$$

(these embeddings will be proved in the proof of Theorem 4b).

To obtain (D1) and (D2) we will need to assume now that

$$\sum_{y\in\mathbb{Z}^d} |y|^{8D}\Gamma(x,z,\{y\}) \leq C\left(1+|x|^{8D}+|z|^{8D}\right) \qquad \text{and} \tag{4.5a}$$

$$\sum_{y_1,y_2\in\mathbb{Z}^d} \left(|y_1|^{8D}+|y_2|^{8D}\right)\Lambda(x_1,x_2,z,\{(y_1,y_2)\}) \leq C\left(1+|x_1|^{8D}+|x_2|^{8D}+|z|^{8D}\right) \tag{4.5b}$$

for all $x_1,x_2,z\in\mathbb{Z}^d$ (the other six inequalities in (D1) and (D2) follow from these two and Jensen's inequality). We remark that in Méléard (1998) the author also needs to assume moments of order $8D$ for the jump rates ($8D+2$ in her case, see (H_1') in her paper).

Theorem 4b. *In the above context, suppose that Assumption C holds and that* (4.2), (4.5a), *and* (4.5b) *hold. Then the conclusion of Theorem 2 is valid, i.e.,* σ_t^N *converges in distribution in* $D([0,T],\ell^{-2,D}(\mathbb{Z}^d))$ *(where* $\ell^{-2,D}(\mathbb{Z}^d)$ *is the dual of* $\ell^{2,D}(\mathbb{Z}^d)$*) to the unique solution* σ_t *of the (*$\ell^\infty(\mathbb{Z}^d)'$*-valued) system given in* (S2).

We recall that the dual of $\ell^\infty(\mathbb{Z}^d)$ can be identified with the space of finitely additive measures on $\mathbb{Z}^d$, and thus every $\xi\in\ell^\infty(\mathbb{Z}^d)'$ can be represented as $\left(\xi(x)\right)_{x\in\mathbb{Z}^d}$ and we can write

$$\langle\xi,\varphi\rangle = \sum_{x\in\mathbb{Z}^d}\varphi(x)\xi(x)$$

for $\varphi\in\ell^\infty(\mathbb{Z}^d)$. Therefore, (S2) can be expressed in this case in a manner analogous to (S2-f).

Example 4.4. Here we consider a well-known model in mathematical biology, the Fleming-Viot process, which was originally introduced in Fleming and Viot (1979) as a stochastic model in population genetics with a constant number of individuals which keeps track of the positions of the individuals. We will actually consider the version of this model studied in Ferrari and Marić (2007).

We take as a type space $W = \mathbb{Z}^+$ and consider an infinite matrix $Q = (q(i,j))_{i,j\in W\cup\{0\}}$ corresponding to the transition rates of a conservative continuous-time Markov process on $W\cup\{0\}$, for which 0 is an absorbing state (observe that, in particular, $q(i,i) = -\sum_{j\neq i} q(i,j)$). Each individual moves independently according to Q, until it gets absorbed at 0. On absorption, it chooses an individual uniformly from the population and jumps (instantaneously) to its position. We assume that the exit rates from each site are uniformly bounded, i.e., $\sup_{i\geq 1}\sum_{j\in(W\cup\{0\})\setminus\{i\}} q(i,j) < \infty$ (this is so that (A1) is satisfied). The rates take the following form:

$$\Gamma(i,k,\{j\}) = \begin{cases} q(i,j) & \text{if } k\neq j \text{ and } i\neq j\\ q(i,j)+q(i,0) & \text{if } k=j \text{ and } i\neq j\\ 0 & \text{if } i=j\end{cases} \qquad \text{and} \qquad \Lambda\equiv 0.$$

Observe that with this definition, the total rate at which a particle at i jumps to j when the whole population is at state ν is given by $q(i,j)+q(i,0)\nu(\{j\})$.

We will write $u_t^N(i) = \nu_t^N(\{i\})$. It is clear that this model satisfies the assumptions of Theorem 1. Therefore, if the initial distributions u_0^N converge, and we denote by u_t the limit given by Theorem 1, we obtain that for each $i\geq 1$,

$$\frac{du_t(\{i\})}{dt} = \sum_{j\geq 1}\big[q(i,j)+q(i,0)u_t(j)\big]u_t(i).$$

This limit was obtained in Theorem 1.2 of Ferrari and Marić (2007) (though there the convergence is proved for each fixed t).

To study the fluctuations process we need to add the following moment assumption on Q:

$$\sum_{j\geq 1} j^8 q(i,j) \leq C(1+i^8)$$

for some $C>0$ independent of i. With this, if (4.2) holds, we can apply Theorem 2. By the remark following Theorem 4b, to express the limiting system for the fluctuations process it is enough to apply (S2-w) to functions of the form $\varphi = \mathbf{1}_i$ for each $i\geq 1$. Doing this, and after some algebraic manipulations, we deduce that the limiting fluctuations process σ_t is the unique process with paths in $C([0,T],\ell^\infty(\mathbb{Z}^+)')$ satisfying the following stochastic differential equation:

$$d\sigma_t = Q^{\mathrm{t}}\sigma_t\,dt + \left(\sum_{k\geq 1} Q(k,0)\sigma_t(k)\right)u_t\,dt + \left(\sum_{k\geq 1} Q(k,0)u_t(k)\right)\sigma_t\,dt + \sqrt{V_t}\,dB_t,$$

where B_t is an infinite vector of independent standard Brownian motions and V_t is given by

$$V_t(i,j) = \begin{cases} \sum_{k\neq i}\big[q(k,i)+q(k,0)u_t(i)\big]u_t(k) - \big[q(i,i)-q(i,0)\big]u_t(i) + q(i,0)u_t(i)^2 & \text{if } i=j,\\ -q(i,j)u_t(i) - q(j,i)u_t(j) - \big[q(i,0)+q(j,0)\big]u_t(i)u_t(j) & \text{if } i\neq j.\end{cases}$$

4.3.3 $W = \Omega$, a compact, sufficiently smooth subset of $\mathbb{R}^d$

Unlike the last case, the type space W is now compact, so we can simply take $\rho_i \equiv 1$. Nevertheless, W is not a discrete set now, and this leads us to use Sobolev spaces for our sequence of continuous embeddings:

$$\mathcal{C}^{3D}(\Omega) \hookrightarrow H^{3D}(\Omega) \underset{c}{\hookrightarrow} H^{2D}(\Omega) \hookrightarrow \mathcal{C}^{D}(\Omega) \hookrightarrow H^{D}(\Omega) \hookrightarrow \mathcal{C}(\Omega) \hookrightarrow L^2(\Omega)$$

(with $D = \lfloor d/2 \rfloor + 1$ as before), where $\mathcal{C}^k(\Omega)$ is the space of continuous functions on Ω with k continuous derivatives, endowed with the norm

$$\|\varphi\|_{\mathcal{C}^k(\Omega)} = \sum_{|\alpha| \leq k} \sup_{x \in \Omega} |\partial^\alpha \varphi(x)|,$$

and $H^k(\Omega)$ is the Sobolev space (with respect to the $L^2(\Omega)$ norm) of order k, i.e., the space of functions on Ω with k weak derivatives in $L^2(\Omega)$, endowed with the norm

$$\|\varphi\|_{H^k(\Omega)}^2 = \sum_{|\alpha| \leq k} \int_\Omega |\partial^\alpha \varphi(x)|^2 \, dx.$$

The above embeddings are either direct or are consequences of the usual Sobolev embedding theorems, see Theorem 4.12 of Adams (2003). For these to hold we need Ω to be sufficiently smooth (a locally Lipschitz boundary is enough). The compact embedding $H^{2D}(\Omega) \hookrightarrow H^D(\Omega)$ is a consequence of the Rellich–Kondrakov Theorem (see Theorem 6.3 of Adams (2003)).

In this case (D1) and (D2) hold trivially. (D3) is much more delicate, and we will just leave it stated as it is. (The assumptions (H3), (H3)$'$, and (H3)$''$ of Méléard (1998) give some particular conditions which, if translated to our setting, would assure that (D3) holds. These conditions are suitable in her setting but they unfortunately rule out some interesting examples for us).

Theorem 4c. *In the above context, assume that Assumption and C holds, and that (D3) and (4.2) hold. Then the conclusion of Theorem 2 is valid, i.e., σ_t^N converges in distribution in $D([0,T], H^{-3D}(\Omega))$ (where $H^{-3D}(\Omega)$ is the dual of $H^{3D}(\Omega)$) to the unique solution σ_t of the ($(\mathcal{C}^{3D}(\Omega))'$-valued) system given in (S2).*

4.3.4 $W = \mathbb{R}^d$

This case combines both of the difficulties encountered before: W is neither discrete nor compact. To get around these problems we need to use now weighted Sobolev spaces. The weighting functions ρ_i are given by

$$\rho_i(x) = \sqrt{1 + |x|^{2iD+2q}},$$

where $D = \lfloor d/2 \rfloor + 1$ and $q \in \mathbb{N}$ (to be chosen). We consider now the spaces $\mathcal{C}^{j,k}$ of continuous functions φ with continuous partial derivatives up to order j and such that

$\lim_{|x|\to\infty} |\partial^\alpha \varphi(x)| / (1+|x|^k) = 0$ for all $|\alpha| \leq j$, with the norms

$$\|\varphi\|_{\mathcal{C}^{j,k}} = \sum_{|\alpha|\leq j} \sup_{x\in\mathbb{R}^d} \frac{|\partial^\alpha \varphi(x)|}{1+|x|^k},$$

(as in Section 4.3.2, the weigthing functions ρ_i do not appear explicitly here, but the definition does not change if we replace the term $1+|x|^k$ by $\sqrt{1+|x|^{2k}}$) and the weighted Sobolev spaces $W_0^{j,k}$ (with respect to the L^2 norm) defined as follows: we define the norms

$$\|\varphi\|^2_{W_0^{j,k}} = \sum_{|\alpha|\leq j} \int_{\mathbb{R}^d} \frac{|\partial^\alpha \varphi(x)|^2}{1+|x|^{2k}}\, dx$$

and let $W_0^{j,k}$ be the closure in L^2 under this norm of the space of functions of class $\mathcal{C}^\infty$ with compact support.

The right sequence of embeddings is now the following:

$$\mathcal{C}^{3D,q} \hookrightarrow W_0^{3D,D+q} \underset{\mathrm{c}}{\hookrightarrow} W_0^{2D,2D+q} \hookrightarrow C^{D,2D+q} \hookrightarrow W_0^{D,3D+q} \hookrightarrow \mathcal{C}^{0,3D+q} \hookrightarrow W_0^{0,4D+q}.$$

$q \in \mathbb{N}$ can be chosen depending on the specific example being analyzed: $q=0$ works for many examples, but as we will see in the next example, choosing a positive q ($q=1$ in that case) can help, for instance, by making all constant functions be in $\mathcal{C}^{3D,q}$. These embeddings are, as before, either straightforward or consequences of the usual Sobolev embedding theorems (adapted now to the weighted case; see Méléard (1998), where the author uses the same type of embeddings, and see Kufner (1980) for a general discussion of weighted Sobolev spaces).

To obtain (D1) and (D2) we need to add the following moment assumptions on the rates, analogous to those we used in Theorem 4b: for all $x, x_1, x_2, z \in \mathbb{R}^d$,

$$\int_{\mathbb{R}^d} |y|^{8D+2q}\, \Gamma(x,z,dy) \leq C\left(1+|x|^{8D+2q}+|z|^{8D+2q}\right) \qquad \text{and} \tag{4.6a}$$

$$\begin{aligned}\int_{\mathbb{R}^d\times\mathbb{R}^d} \left(|y_1|^{8D+2q}+|y_2|^{8D+2q}\right)\, \Lambda(x_1,x_2,z,dy_1\otimes dy_2)& \\ \leq C\left(1+|x_1|^{8D+2q}+|x_2|^{8D+2q}+|z|^{8D+2q}\right).&\end{aligned} \tag{4.6b}$$

We observe that the power $8D+2q$ appearing in this assumption corresponds exactly, when $q=1$, to the moments of order $8D+2$ assumed in (H_1') in Méléard (1998). (D3), as in the previous case, is much more involved, so we will again leave it stated as it is.

Theorem 4d. *In the above context, assume moreover that Assumption C holds, and that (4.2), (D3), (4.6a), and (4.6b) hold. Then the conclusion of Theorem 2 is valid, i.e., σ_t^N converges in distribution in $D([0,T], W_0^{-3D,D+q})$ (where $W_0^{-3D,D+q}$ is the dual of $W_0^{3D,D+q}$) to the unique solution σ_t of the ($(\mathcal{C}^{3D,q})'$-valued) system given in (S2).*

Example 4.5 (Continuation of Example 3.3)**.** In the previous section we obtained the system (3.1) that characterizes the information percolation model of Duffie and Manso (2007) by using (S1′). If we use (S1) instead we obtain

$$\frac{d}{dt}\langle \nu_t, \varphi\rangle = 2\lambda \langle \nu_t, \nu_t * \varphi\rangle - 2\lambda\langle \nu_t, \varphi\rangle$$

for all $\varphi \in \mathcal{B}(\mathbb{R})$, where $(\nu_s * \varphi)(z) = \int_W \varphi(x+z)\,\nu_s(dx)$.

To obtain the fluctuations limit, we need to check the assumptions of Theorem 4d. As we mentioned, we will take $q = 1$. Assumption C holds trivially because $\lambda(w_1, w_2, \nu)$ and $b(w_1, w_2, \nu, \cdot)$ do not depend on ν. We will assume that the initial distribution of the system satisfies (4.2). (4.6a) and (4.6b) are straightforward to check in this case.

We are left checking (D3). Let $\varphi \in \mathcal{C}^{3,1}$ and take $\mu_1, \mu_2, \mu_3, \mu_4 \in \mathcal{P}$ having moments of order 10. We have that

$$J_{\mu_1,\mu_2}\varphi(z) = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} \big(2\varphi(w_1+w_2) - \varphi(w_1) - \varphi(w_2)\big)\,\mu_1(dw_2)\,\mu_1(dw_1) + \int_{-\infty}^{\infty} \big(2\varphi(w+z) - \varphi(w) - \varphi(z)\big)\,\big[\mu_1(dw) + \mu_2(dw)\big].$$

The first term on the right side is constant in z, so it is in $\mathcal{C}^{3,1}$ (this is why we needed $q = 1$ in this example). For the second term, since $|\varphi(x)| \leq C\,\|\varphi\|_{\mathcal{C}^{3,1}}\,(1+|x|)$ and $\big\langle \mu_i, 1 + |\cdot|^{10} \big\rangle < \infty$, the integral is bounded, and hence the derivatives with respect to z can be taken inside the integral, whence we get that this term is also in $\mathcal{C}^{3,1}$. The same argument can be repeated for $\mathcal{C}^{1,3}$ and $\mathcal{C}^{0,4}$. The fact that the norm of this operator in these spaces is bounded uniformly in μ_1, μ_2 follows from the same argument. This gives (D3.i). Using the same formula it is easy to show that

$$\big\|\big(J_{\mu_1,\mu_2} - J_{\mu_3,\mu_4}\big)\varphi\big\|_{\mathcal{C}^{3,1}} \leq C\,\|\varphi\|_{\mathcal{C}^{3,1}}\Big[\,\|\mu_1 - \mu_3\|_{(\mathcal{C}^{3,1})'} + \|\mu_2 - \mu_4\|_{(\mathcal{C}^{3,1})'}\,\Big],$$

which is stronger than (D3.ii).

We have checked all the assumptions of Theorem 4d, so we deduce that the fluctuations process σ_t^N converges in distribution in $W_0^{-3,2}$ to the unique solution of (S2) (which is an equation in $(\mathcal{C}^{3,1})'$). Writing down the formula for J_s in this case yields

$$\langle \sigma_s, J_s\varphi\rangle = 4\lambda\,\langle \sigma_s, \nu_s * \varphi\rangle - 2\lambda\,\langle \sigma_s, \varphi\rangle$$

for every $\varphi \in \mathcal{C}^{3,1}$. For the quadratic covariations we get

$$C_s^{\varphi_1,\varphi_2} = 4\lambda\,\langle \nu_s, \nu_s * (\varphi_1\varphi_2)\rangle - 6\lambda\,\langle \nu_s, \varphi_1\rangle\,\langle \nu_s, \varphi_2\rangle + 2\lambda\,\langle \nu_s, \varphi_1\varphi_2\rangle$$

for every $\varphi_1, \varphi_2 \in \mathcal{C}^{3,1}$. Therefore the limiting fluctuations satisfy

$$\langle \sigma_t, \varphi\rangle = \langle \sigma_0, \varphi\rangle + \lambda\int_0^t \left[4\,\langle \sigma_s, \nu_s * \varphi\rangle - 2\,\langle \sigma_s, \varphi\rangle\right] ds + Z_t(\varphi),$$

with Z_t being a centered Gaussian process taking values in the dual of $\mathcal{C}^{3,1}$ with quadratic covariations given by $[Z(\varphi_1), Z(\varphi_2)]_t = \int_0^t C_s^{\varphi_1,\varphi_2}\,ds$ for each $\varphi_1, \varphi_2 \in \mathcal{C}^{3,1}$.

5 Proofs of the Results

Throughout this section, C, C_1, and C_2 will denote constants whose values might change from line to line.

5.1 Preliminary computations and proof of Theorem 1

Since ν_t^N is a jump process in $\mathcal{P}$ with bounded jump rates, its generator is given by

$$
\begin{aligned}
\Omega_N f(\nu) = N \int_W \gamma(w,\nu) \int_W \Delta_N f(\nu; w; w')\, a(w,\nu,dw')\, \nu(dw) \\
+ N \int_W \int_W \lambda(w_1,w_2,\nu) \int_{W\times W} \Delta_N f(\nu; w_1, w_2; w_1' w_2') \\
\cdot\, b(w_1,w_2,\nu,dw_1 \otimes dw_2'))\, \nu(dw_1)\, \nu(dw_2)
\end{aligned} \tag{5.1}
$$

for any bounded measurable function f on $\mathcal{P}$, where $\Delta_N f(\nu;w;w') = f\big(\nu + N^{-1}(\delta_{w'} - \delta_w)\big) - f(\nu)$ and $\Delta_N f(\nu; w_1, w_2; w_1', w_2') = f\big(\nu + N^{-1}(\delta_{w_1'} + \delta_{w_2'} - \delta_{w_1} - \delta_{w_2})\big) - f(\nu)$.

Given $\varphi \in \mathcal{B}(W)$ we get by using (5.1) and Proposition IV.1.7 of Ethier and Kurtz (1986) for $f(\nu) = \langle \nu, \varphi\rangle$ that

$$
\begin{aligned}
\big\langle \nu_t^N, \varphi \big\rangle = \big\langle \nu_0^N, \varphi \big\rangle + M_t^{N,\varphi} + \int_0^t \int_W \gamma(w,\nu_s^N) \int_W (\varphi(w') - \varphi(w))\, a(w,\nu_s^N,dw')\, \nu_s^N(dw)\, ds \\
+ \int_0^t \int_W \int_W \lambda(w_1,w_2,\nu_s^N) \int_{W\times W} (\varphi(w_1') + \varphi(w_2') - \varphi(w_1) - \varphi(w_2)) \\
\cdot\, b(w_1,w_2,\nu_s^N, dw_1' \otimes dw_2)\, \nu_s^N(dw_2)\, \nu_s^N(dw_1)\, ds,
\end{aligned} \tag{5.2}
$$

where $M_t^{N,\varphi}$ is a martingale starting at 0. This formula is the key to the proof of Theorem 1 because, ignoring the martingale term, this equation has the exact form we need for obtaining (S1), and thus the idea will be to show that $M_t^{N,\varphi}$ vanishes in the limit as $N \to \infty$. This follows from the fact that the quadratic variation of $M_t^{N,\varphi}$ is of order $O(1/N)$. More precisely, we have the following formula: for any $\varphi_1, \varphi_2 \in \mathcal{B}(W)$, the predictable quadratic covariation between the martingales M_t^{N,φ_1} and M_t^{N,φ_2} is given by

$$
\begin{aligned}
\Big\langle M^{N,\varphi_1}, M^{N,\varphi_2} \Big\rangle_t = \frac{1}{N} \int_0^t \int_W \gamma(w,\nu_s^N) \int_W (\varphi_1(w') - \varphi_1(w))(\varphi_2(w') - \varphi_2(w)) \\
\cdot\, a(w,\nu_s^N,dw')\, \nu_s^N(dw)\, ds \\
+ \frac{1}{N} \int_0^t \int_W \int_W \lambda(w_1,w_2,\nu_s^N) \int_{W\times W} (\varphi_1(w_1') + \varphi_1(w_2') - \varphi_1(w_1) - \varphi_1(w_2)) \\
\cdot\, (\varphi_2(w_1') + \varphi_2(w_2') - \varphi_2(w_1) - \varphi_2(w_2)) \\
\cdot\, b(w_1,w_2,\nu_s^N, dw_1' \otimes dw_2')\, \nu_s^N(dw_2)\, \nu_s^N(dw_1)\, ds.
\end{aligned} \tag{5.3}
$$

The proof of this formula is almost the same as that of Proposition 3.4 of Fournier and Méléard (2004) so we will omit it (there the computation is done for $\varphi_1 = \varphi_2$, but the generalization is straightforward, and can also be obtained by polarization).

Proof of Theorem 1. The proof is relatively standard, and its basic idea is the following. First one proves that the sequence of processes $\big\langle \nu_t^N, \varphi \big\rangle$ is tight in $D([0,T],\mathbb{R})$ for each $\varphi \in \mathcal{C}_b(W)$, which in turn implies the tightness of ν_t^N in $D([0,T],\mathcal{P})$. The tightness of these processes follows from standard techniques and uses (5.2) and (5.3). Next, one uses a martingale argument and (5.3) to show that any limit point of ν_t^N satisfies (S1). Finally, using Gronwall's Lemma one deduces that (S1) has a unique solution. We refer the reader to the proof of Theorem 5.3 of Fournier and Méléard (2004) for the details. □

Proof of Corollary 3.1. Denote by $(\tau_i^N)_{i>0}$ the sequence of stopping times corresponding to the jumps of the process ν_t^N. Let A be any Borel subset of W with $\mu(A) = 0$ and let φ be any positive function in $\mathcal{B}(W)$ whose support is contained in A. By (5.2), for every $t \in [0, T]$ we have that

$$\begin{aligned}\mathbb{E}\left(\left\langle \nu^N_{t\wedge\tau_1^N}, \varphi\right\rangle\right) &= \mathbb{E}\left(\langle \nu_0, \varphi\rangle\right) + \mathbb{E}\left(M^{N,\varphi}_{t\wedge\tau_1^N}\right)\\ &\quad + \mathbb{E}\left(\int_0^{t\wedge\tau_1^N}\int_W \gamma(w,\nu_s^N)\int_W (\varphi(w') - \varphi(w))\, a(w,\nu_s^N,dw')\,\nu_s^N(dw)\,ds\right)\\ &\quad + \mathbb{E}\Bigg(\int_0^{t\wedge\tau_1^N}\int_W\int_W \lambda(w_1,w_2,\nu_s^N)\int_{W\times W}(\varphi(w_1') + \varphi(w_2') - \varphi(w_1) - \varphi(w_2))\\ &\qquad\qquad \cdot\, b(w_1,w_2,\nu_s^N,dw_1'\otimes dw_2'))\,\nu_s^N(dw_2)\,\nu_s^N(dw_1)\,ds\Bigg).\end{aligned} \tag{5.4}$$

The first term on the right side of (5.4) is 0 because the support of φ is contained A and $\nu_0(A) = 0$. The second term is 0 by Doob's Optional Sampling Theorem. For the third term observe that for $s < \tau_1^N$, $\nu_s^N = \nu_0$, so

$$\begin{aligned}\mathbb{E}\left(\left|\int_0^{t\wedge\tau_1^N}\int_W \gamma(w,\nu_s^N)\int_W (\varphi(w') - \varphi(w))\, a(w,\nu_s^N,dw')\,\nu_s^N(dw)\,ds\right|\right)&\\ \leq \overline{\gamma}\mathbb{E}\left(\int_0^{t\wedge\tau_1^N}\int_W\int_W |\varphi(w') - \varphi(w)|\, a(w,\nu_0,dw')\,\nu_0(dw)\right)&\end{aligned}$$

which is 0 since φ is supported inside A and the measure $\int_W a(w',\nu_0,\cdot)\,\nu_0(dw')$ is absolutely continuous with respect to μ. The fourth term is 0 by analogous reasons. We deduce that the expectation on the left side of (5.4) is 0, and therefore, since φ is positive, $\left\langle \nu^N_{t\wedge\tau_1^N}, \varphi\right\rangle = 0$ with probability 1. In particular, $\nu^N_{t\wedge\tau_1^N}$ is absolutely continuous with respect to μ for all $t \in [0, T]$ with probability 1.

Using the strong Markov property we obtain inductively that $\left\langle \nu^N_{t\wedge\tau_i^N}, \varphi\right\rangle = 0$ almost surely for every $i > 0$ and $t \in [0, T]$. Since the jump rates of the process are bounded, there are finitely many jumps before T with probability 1, and we deduce that $\left\langle \nu_t^N, \varphi\right\rangle = 0$ almost surely for all $t \in [0, T]$. Now if ν_t is the limit in distribution of the sequence ν_t^N given by Theorem 1 and $\varphi \in \mathcal{C}_b(W)$, $\mathbb{E}\left(\left\langle \nu_t^N, \varphi\right\rangle\right) \to \langle \nu_t, \varphi\rangle$ as $N \to \infty$, so $\langle \nu_t, \varphi\rangle = 0$ for all $t \in [0, T]$ whenever φ is supported inside A, and the result follows. □

5.2 Proof of Theorem 2

We will assume throughout this part that all the assumptions of Theorem 2 hold. For simplicity, we will also assume that $\Gamma \equiv 0$ (these terms are easier to handle and are in fact a particular case of the ones corresponding to Λ).

Before getting started we recall that, by Parseval's identity, given any $A \in \mathcal{H}_i{}'$ and a complete orthonormal basis $(\phi_k)_{k\geq 0}$ of $\mathcal{H}_i$,

$$\|A\|^2_{\mathcal{H}_i{}'} = \sum_{k\geq 0} |A(\phi_k)|^2.$$

We will use this fact several times below.

5.2.1 Moment estimates for ν_t^N and ν_t

Recall that we have assumed that $\sup_{N>0} \mathbb{E}\big(\langle \nu_0^N, \rho_4^2\rangle\big) < \infty$ and $\mathbb{E}\big(\langle \nu_0, \rho_4^2\rangle\big) < \infty$. We need to show that these moment assumptions propagate to ν_t^N and ν_t:

Proposition 5.1. *The following properties hold:*

$$\sup_{N>0} \mathbb{E}\left(\sup_{t\in[0,T]} \langle \nu_t^N, \rho_4^2\rangle\right) < \infty, \quad \textit{and} \tag{5.5a}$$

$$\sup_{t\in[0,T]} \langle \nu_t, \rho_4^2\rangle < \infty. \tag{5.5b}$$

The proof of this result will rely on an explicit construction of the process in terms of Poisson point measures. This is similar to what is done in Section 2.2 of Fournier and Méléard (2004) (though we will need to use a more abstract approach because our type spaces are not necessarily Euclidean), so we will only sketch the main ideas.

We fix $N > 0$ and consider the following random objects, defined on a sufficiently large probability space: a $\mathcal{P}$-valued random variable ν_0^N (corresponding to the initial distribution) and a Poisson point measure $Q(ds, di, dj, du, d\theta)$ on $[0,T]\times I_N\times I_N\times[0,1]\times[0,1]$ with intensity measure $(\overline{\lambda}/N)\, ds\, di\, dj\, du\, d\theta$. We also consider a Blackwell-Dubins representation ϱ of $\mathcal{P}(W\times W)$ with respect to a uniform random variable on $[0,1]$, i.e., a continuous function $\varrho\colon \mathcal{P}(W\times W)\times[0,1] \longrightarrow W\times W$ such that $\varrho(\xi,\cdot)$ has distribution ξ (with respect to the Lebesgue measure on $[0,1]$) for all $\xi \in \mathcal{P}(W\times W)$ and $\varrho(\cdot,u)$ is continuous for almost every $u\in[0,1]$ (see Blackwell and Dubins (1983) for the existence of such a function). This gives us an abstract way to use a uniform random variable to pick the pairs of types to which agents go after interacting. Finally, we introduce the following notation: $\eta^i(\nu_t^N)$ will denote the i-th type, with respect to some fixed total order of W, appearing in ν_t^N (we recall that, under the axiom of choice, any set can be well-ordered, and hence totally ordered; moreover, this ordering can be taken to be measurable because W, being a Polish space, is measurably isomorphic to $[0,1]$). With this definition, choosing a type uniformly from ν_t^N is the same as choosing i uniformly from I_N and considering the type given by $\eta^i(\nu_t^N)$. Our process can be represented then as follows:

$$\begin{aligned}\nu_t^N = \nu_0^N + \int_0^t\int_{I_N}\int_{I_N}\int_0^1\int_0^1 \frac{1}{N}\Big[&\delta_{\varrho^1(b(\eta^i(\nu_{s-}^N),\eta^j(\nu_{s-}^N),\nu_{s-}^N,\cdot),u)}\\ &+\delta_{\varrho^2(b(\eta^i(\nu_{s-}^N),\eta^j(\nu_{s-}^N),\nu_{s-}^N,\cdot),u)} - \delta_{\eta^i(\nu_{s-}^N)} - \delta_{\eta^j(\nu_{s-}^N)}\Big]\\ &\cdot \mathbf{1}_{\theta\leq\lambda(\eta^i(\nu_{s-}^N),\eta^j(\nu_{s-}^N),\nu_{s-}^N)/\overline{\lambda}}\; Q(ds,di,dj,du,d\theta),\end{aligned}$$

where ϱ^1 and ϱ^2 are the first and second components of ϱ (see Definition 2.5 in Fournier and Méléard (2004) for more details on this construction).

Proof of Proposition 5.1. Since $\langle \nu_t^N, \rho_4^2\rangle = \langle \nu_0^N, \rho_4^2\rangle + \sum_{0\leq s\leq t}\big[\langle \nu_s^N - \nu_{s-}^N, \rho_4^2\rangle\big]$, it is easy to deduce from the last equation that

$$\begin{aligned}\langle \nu_t^N, \rho_4^2\rangle = \langle \nu_0^N, \rho_4^2\rangle + \int_0^t\int_{I_N}\int_{I_N}\int_0^1\int_0^1 \frac{1}{N}\Big[&\rho_4^2(\varrho^1(b(\eta^i(\nu_{s-}^N),\eta^j(\nu_{s-}^N),\nu_{s-}^N,\cdot),u))\\ &+\rho_4^2(\varrho^2(b(\eta^i(\nu_{s-}^N),\eta^j(\nu_{s-}^N),\nu_{s-}^N,\cdot),u)) - \rho_4^2(\eta^i(\nu_{s-}^N)) - \rho_4^2(\eta^j(\nu_{s-}^N))\Big]\\ &\cdot \mathbf{1}_{\theta\leq\lambda(\eta^i(\nu_{s-}^N),\eta^j(\nu_{s-}^N),\nu_{s-}^N)/\overline{\lambda}}\; Q(ds,di,dj,du,d\theta).\end{aligned}$$

Taking expectations and ignoring the (positive) terms being subtracted we obtain

$$\begin{aligned}\mathbb{E}\Bigg(\sup_{t\in[0,T]}\langle \nu_t^N, \rho_4^2\rangle\Bigg) &\leq \mathbb{E}\big(\langle \nu_0^N, \rho_4^2\rangle\big) + \frac{1}{N^2}\int_0^T \mathbb{E}\Bigg(\sum_{i=1}^N\sum_{j=1}^N \lambda(\eta^i(\nu_{s-}^N),\eta^j(\nu_{s-}^N),\nu_{s-}^N)\\ &\qquad\cdot\int_0^1\Big[\rho_4^2(\varrho^1(b(\eta^i(\nu_{s-}^N),\eta^j(\nu_{s-}^N),\nu_{s-}^N,\cdot),u))\\ &\qquad+\rho_4^2(\varrho^2(b(\eta^i(\nu_{s-}^N),\eta^j(\nu_{s-}^N),\nu_{s-}^N,\cdot),u))\Big]\,du\Bigg)\,ds\\ &= \mathbb{E}\big(\langle \nu_0^N, \rho_4^2\rangle\big) + \int_0^T \mathbb{E}\Bigg(\int_W\int_W\int_W\int_{W\times W}\big[\rho_4^2(w_1') + \rho_4^2(w_2')\big]\,\Lambda(w_1,w_2,z,dw_1'\otimes dw_2')\\ &\qquad\cdot \nu_s^N(dz)\,\nu_s^N(dw_2)\,\nu_s^N(dw_1)\Bigg)\,ds\\ &\leq \mathbb{E}\big(\langle \nu_0^N, \rho_4^2\rangle\big) + C\int_0^T \mathbb{E}\Bigg(\int_W\int_W\int_W\big[\rho_4^2(w_1) + \rho_4^2(w_2) + \rho_4^2(z)\big]\\ &\qquad\cdot \nu_s^N(dz)\,\nu_s^N(dw_2)\,\nu_s^N(dw_1)\Bigg)\,ds\\ &\leq \mathbb{E}\big(\langle \nu_0^N, \rho_4^2\rangle\big) + C\int_0^T \mathbb{E}\Bigg(\sup_{s\in[0,t]}\langle \nu_s^N, \rho_4^2\rangle\Bigg)\,ds,\end{aligned}$$

where we used (D2) in the second inequality. By hypothesis $\mathbb{E}\big(\langle \nu_0^N, \rho_4^2\rangle\big)$ is bounded uniformly in N, so by Gronwall's Lemma we deduce that

$$\mathbb{E}\Bigg(\sup_{t\in[0,T]}\langle \nu_t^N, \rho_4^2\rangle\Bigg) \leq C_1 e^{C_2 T}, \tag{5.6}$$

with C_1 and C_2 being independent of N, whence (5.5a) follows.

To get (5.5b), write $(\rho_4^2\wedge L)(w) = \rho_4^2(w)\wedge L$, and observe that, since $\rho_4^2\wedge L\in\mathcal{C}_b(W)$, Theorem 1 implies that $\lim_{N\to\infty}\mathbb{E}\Big(\sup_{t\in[0,T]}\langle \nu_t^N, \rho_4^2\wedge L\rangle\Big) = \sup_{t\in[0,T]}\langle \nu_t, \rho_4^2\wedge L\rangle$, so by

(5.6),

$$\sup_{t\in[0,T]} \left\langle \nu_t, \rho_4^2 \wedge L \right\rangle \leq C_1 e^{C_2 T}.$$

Using the Monotone Convergence Theorem it is easy to check that $\sup_{s\in[0,T]} \left\langle \nu_s, \rho_4^2 \wedge L \right\rangle \to \sup_{s\in[0,T]} \left\langle \nu_s, \rho_4^2 \right\rangle$ as $L \to \infty$, and thus (5.5b) follows. □

For most of this section we will continue ignoring the type-process η_t^N, working instead with the empirical distribution process ν_t^N we are interested in. However, we will need to consider η_t^N directly in Step 2 of the proof of Theorem 2, and we will need to use a moment estimate similar to (5.5a) for this process. Observe that statement of the theorem (and that of Theorem 1) makes no assumption on the distribution of η_0^N, but instead only deals with the initial empirical distribution ν_0^N. Therefore we are free to choose η_0^N in any way compatible with ν_0^N. For convenience we can construct η_0^N in the following way: assuming ν_0^N takes a specific value $\overline{\nu}_0^N \in \mathcal{P}_a$, choose $\eta_0^N(1)$ uniformly from $\overline{\nu}_0^N$ and then inductively choose $\eta_0^N(i)$ uniformly from the remaining $N - i + 1$ individuals, i.e., from $\left[N\overline{\nu}_0^N - \delta_{\eta_0^N(1)} - \cdots - \delta_{\eta_0^N(i-1)}\right]/(N - i + 1)$. It is clear then that, with this choice, η_0^N is exchangeable and $\frac{1}{N}\sum_{i=1}^N \delta_{\eta_0^N(i)} = \overline{\nu}_0^N$ as required. Moreover, given any $i \in I_N$, $\mathbb{E}\big(\rho_4^2(\eta_0^N(i))\big) = \mathbb{E}\big(\left\langle \nu_0^N, \rho_4^2 \right\rangle\big)$, and thus the moment assumption that we made on ν_0^N can be rewritten as $\sup_{N>0}\sup_{i\in I_N} \mathbb{E}\big(\rho_4^2(\eta_0^N(i))\big) < \infty$ for all $i \in I_N$. The proof of (5.5a) can then be adapted (by modifying slightly the explicit construction we made of ν_t^N to deal with η_t^N) to obtain

$$\sup_{N>0}\sup_{i\in I_N} \mathbb{E}\left(\sup_{t\in[0,T]} \rho_4^2(\eta_t^N(i))\right) < \infty. \tag{5.7}$$

(We remark that the proof of this estimate uses (5.5a) itself).

5.2.2 Extension of $\left\langle \nu_t^N, \cdot \right\rangle$ and $\left\langle \nu_t, \cdot \right\rangle$ to ${\mathcal{H}_4}'$

The $\mathcal{P}$-valued process ν_t^N can be seen as a linear functional on $\mathcal{B}(W)$ via the mapping $\varphi \longmapsto \left\langle \nu_t^N, \varphi \right\rangle$, and the same can be done for ν_t. However, since $\mathcal{H}_4$ consists of measurable but not necessarily bounded functions, the integrals $\left\langle \nu_t^N, \varphi \right\rangle$ and $\left\langle \nu_t, \varphi \right\rangle$ may diverge. Our first task will be to show that these integrals are finite and, moreover, that ν_t^N (and ν_t) can be seen as taking values in ${\mathcal{H}_4}'$ (and thus also in all the other dual spaces we are considering). A consequence of this will be that σ_t^N is well defined as an ${\mathcal{H}_4}'$-valued process.

Proposition 5.2. *The mapping $\varphi \in \mathcal{H}_4 \mapsto \left\langle \nu_t^N, \varphi \right\rangle$ is in ${\mathcal{H}_4}'$ almost surely for every $t \in [0,T]$ and $N > 0$. Analogously, the mapping $\varphi \in \mathcal{H}_4 \mapsto \left\langle \nu_t, \varphi \right\rangle$ is in ${\mathcal{H}_4}'$ for every $t \in [0,T]$.*

Furthermore, ν_t satisfies (S1) for every $\varphi \in \mathcal{H}_4$, while ν_t^N satisfies (5.2) for every $\varphi \in \mathcal{H}_4$ almost surely. In particular, given any $\varphi \in \mathcal{H}_4$, $M_t^{N,\varphi}$ is a martingale starting at 0 such that the predictable quadratic covariations $\left\langle M^{N,\varphi_1}, M^{N,\varphi_2} \right\rangle_t$ are the ones given by the formula in (5.3) for all $\varphi_1, \varphi_2 \in \mathcal{H}_4$.

Proof. We are only going to prove the assertions for ν_t^N, the ones for ν_t can be checked similarly (and more easily).

The first claim follows directly from (B2) and Proposition 5.1: for $\varphi \in \mathcal{H}_4$,

$$\left|\left\langle \nu_t^N, \varphi \right\rangle\right| \leq \int_W |\varphi(w)| \, \nu_t^N(dw) \leq C \, \|\varphi\|_{\mathcal{H}_4} \int_W \rho_4(w) \, \nu_t^N(dw) \leq C \, \|\varphi\|_{\mathcal{H}_4} \sqrt{\left\langle \nu_t^N, \rho_4^2 \right\rangle},$$

and the term inside the square root is almost surely bounded by (5.5a), so the mapping $\varphi \in \mathcal{H}_4 \longmapsto \left\langle \nu_t^N, \varphi \right\rangle$ is continuous almost surely.

Next we need to show that $\left\langle \nu_t^N, \varphi \right\rangle$ satisfies (5.2) for all $\varphi \in \mathcal{H}_4$. That is, we need to show that the formula

$$M_t^{N,\varphi} = \left\langle \nu_t^N, \varphi \right\rangle - \left\langle \nu_0^N, \varphi \right\rangle - \int_0^t \int_W \int_W \int_W \Lambda\varphi(w_1, w_2; z) \, \nu_s^N(dz) \, \nu_s^N(dw_2) \, \nu_s^N(dw_1) \, ds$$

defines a martingale for each $\varphi \in \mathcal{H}_4$. Let $\varphi \in \mathcal{H}_4$ and $m > 0$ and write $(\varphi \wedge m)(w) = \varphi(w) \wedge m$. $\varphi \wedge m$ is in $\mathcal{B}(W)$, so $M_t^{N,\varphi\wedge m}$ is a martingale. We deduce that given any $0 \leq s_1 \leq \cdots \leq s_k < s < t$ and any continuous bounded functions $\psi_1, \ldots, \psi_k$ on $\mathcal{H}_4$, if we let

$$X^m = \psi_1(\nu_{s_1}^N) \cdots \psi_k(\nu_{s_k}^N) \left[M_t^{N,\varphi\wedge m} - M_s^{N,\varphi\wedge m}\right],$$

then $\mathbb{E}(X^m) = 0$. Using the Monotone Convergence Theorem one can show that $X^m \to \psi_1(\nu_{s_1}^N) \cdots \psi_k(\nu_{s_k}^N) \left[M_t^{N,\varphi} - M_s^{N,\varphi}\right]$ as $m \to \infty$. On the other hand, the sequence $(X^m)_{m>0}$ is uniformly integrable. Indeed, using (B2) and (5.5a) one can show that

$$\mathbb{E}\left(\left|\left(\psi_1(\nu_{s_1}^N) \cdots \psi_k(\nu_{s_k}^N)\left[M_t^{N,\varphi\wedge m} - M_s^{N,\varphi\wedge m}\right]\right|^2\right) \leq Ct^2 \mathbb{E}\left(\sup_{r\in[0,t]} \left\langle \nu_r^N, \rho_4^2 \right\rangle\right) < \infty.$$

We deduce that

$$\mathbb{E}(\psi_1(\nu_{s_1}^N) \cdots \psi_k(\nu_{s_k}^N) \left[M_t^{N,\varphi} - M_s^{N,\varphi}\right]) = \lim_{m\to\infty} \mathbb{E}(X^m) = 0,$$

which implies that $M_t^{N,\varphi}$ is a martingale. The fact that $\left\langle M^{N,\varphi_1}, M^{N,\varphi_2} \right\rangle_t$ has the right form follows from the same arguments as those for (5.3) (here we need to replace φ_1 and φ_2 by φ_1^m and φ_2^m and then take $m \to \infty$ as above). ☐

5.2.3 The drift term

By Proposition 5.2, we have now that the fluctuations process σ_t^N is well defined as a process taking values in ${\mathcal{H}_4}'$ and it satisfies

$$\begin{aligned}\left\langle \sigma_t^N, \varphi \right\rangle &= \sqrt{N} \left\langle \nu_0^N - \nu_0, \varphi \right\rangle + \sqrt{N} M_t^{N\varphi} \\ &\quad + \sqrt{N} \int_0^t \int_W \int_W \int_W \Lambda\varphi(w_1, w_2; z) \left[\nu_s^N(dz)\nu_s^N(dw_2)\nu_s^N(dw_1) - \nu_s(dz)\nu_s(dw_2)\nu_s(dw_1)\right] ds\end{aligned}$$

for every $\varphi \in \mathcal{H}_4$. The integral term can be rewritten as

$$\begin{aligned}\int_0^t \int_W \int_W \int_W \Lambda\varphi(w_1, w_2; z) \Big[&\sigma_s^N(dz)\nu_s^N(dw_2)\nu_s^N(dw_1) \\ &+ \nu_s(dz)\big(\sigma_s^N(dw_2)\nu_s^N(dw_1) + \nu_s(dw_2)\sigma_s^N(dw_1)\big)\Big] ds.\end{aligned}$$

Therefore,

$$\left\langle \sigma_t^N, \varphi \right\rangle = \sqrt{N} \left\langle \nu_0^N - \nu_0, \varphi \right\rangle + \sqrt{N} M_t^{N,\varphi} + \int_0^t \left\langle \sigma_s^N, J_s^N \varphi \right\rangle ds \tag{5.8}$$

for each $\varphi \in \mathcal{H}_4$, where

$$\begin{aligned} J_s^N \varphi(z) = &\int_W \int_W \Lambda\varphi(w_1, w_2; z)\, \nu_s^N(dw_2)\, \nu_s^N(dw_1) \\ &+ \int_W \int_W \Lambda\varphi(w, z; x)\, \nu_s^N(dw)\, \nu_s(dx) + \int_W \int_W \Lambda\varphi(z, w; x)\, \nu_s(dw)\, \nu_s(dx). \end{aligned} \tag{5.9}$$

Observe that $J_s^N = J_{\nu_s^N, \nu_s}$ and $J_s = J_{\nu_s, \nu_s}$, where the operators J_{μ_1, μ_2} are the ones defined in Assumption D. Hence (D3) and Proposition 5.1 imply that J_s^N and J_s are bounded linear operators on each space $\mathcal{C}_i$ $(i = 0, 2, 3)$ and, moreover, for all $\varphi \in \mathcal{C}_i$,

$$\left\| J_s^N \varphi \right\|_{\mathcal{C}_i} \leq C \left\| \varphi \right\|_{\mathcal{C}_i} \qquad \text{and} \qquad \left\| J_s \varphi \right\|_{\mathcal{C}_i} \leq C \left\| \varphi \right\|_{\mathcal{C}_i}, \tag{5.10}$$

almost surely for some constant $C > 0$ independent of N and s. Similarly, given any $\varphi \in \mathcal{C}_0$,

$$\left\| \left(J_s^N - J_s \right) \varphi \right\|_{\mathcal{C}_0} \leq C \left\| \varphi \right\|_{\mathcal{C}_0} \left\| \nu_s^N - \nu_s \right\|_{\mathcal{C}_2{}'} \tag{5.11}$$

almost surely for some constant $C > 0$ independent of N and s.

5.2.4 Uniform estimate for the martingale term in $\mathcal{H}_4{}'$

Proposition 5.2 implies that the martingale term $M_t^{N,\varphi}$ is well defined for all $\varphi \in \mathcal{H}_4$. We will denote by M_t^N the bounded linear functional on $\mathcal{H}_4$ given by $M_t^N(\varphi) = M_t^{N,\varphi}$.

Theorem 5.3. *$\sqrt{N} M_t^N$ is a càdlàg square integrable martingale in $\mathcal{H}_4{}'$, whose Doob–Meyer process $\left(\langle\!\langle \sqrt{N} M^N \rangle\!\rangle_t (\varphi_1) \right)(\varphi_2) = N \left\langle \sqrt{N} M^N(\varphi_1), \sqrt{N} M^N(\varphi_2) \right\rangle_t$ (which is a linear operator from $\mathcal{H}_4$ to $\mathcal{H}_4{}'$) can be obtained from the formula in (5.3). Moreover,*

$$\sup_{N>0} \mathbb{E} \left(\sup_{t \in [0,T]} \left\| \sqrt{N} M_t^N \right\|_{\mathcal{H}_4{}'}^2 \right) < \infty.$$

Proof. We already know, by Proposition 5.2, that $\sqrt{N} M_t^N$ is a martingale in $\mathcal{H}_4{}'$ with the right Doob–Meyer process. The fact that the paths of $\sqrt{N} M_t^N$ are in $D([0,T], \mathcal{H}_4{}')$ can be checked by the same arguments as those in the proof of Corollary 3.8 in Méléard (1998). So we only need to show the last assertion. Let $(\phi_k)_{k \geq 0}$ be an orthonormal complete basis of $\mathcal{H}_4$. We observe that, by (B2), if $\chi_w \in \mathcal{H}_4{}'$ is defined by $\chi_w(\varphi) = \varphi(w)$ then

$$\sum_{k \geq 0} \phi_k^2(w) = \left\| \chi_w \right\|_{\mathcal{H}_4{}'}^2 \leq C \rho_4^2(w).$$

Thus by Proposition 5.2 and Doob's inequality,

$$\begin{aligned}
\mathbb{E}\Bigg(\sup_{t\in[0,T]}\left\|\sqrt{N}M_t^N\right\|_{\mathcal{H}_4{}'}^2\Bigg) &\leq \mathbb{E}\Bigg(\sum_{k\geq 0}\sup_{t\in[0,T]} N\left|M_t^{N,\phi_k}\right|^2\Bigg) \leq 4\sum_{k\geq 0}\mathbb{E}\big(N\langle M^{N,\phi_k},M^{N,\phi_k}\rangle_T\big)\\
&= 4\mathbb{E}\Bigg(\int_0^T\int_W\int_W\int_W\int_{W\times W}\sum_{k\geq 0}\big(\phi_k(w_1')-\phi_k(w_1)+\phi_k(w_2')-\phi_k(w_2)\big)^2\\
&\qquad\qquad\cdot\Lambda(w_1,w_2,z,dw_1'\otimes dw_2')\,\nu_s^N(dz)\,\nu_s^N(dw_2)\,\nu_s^N(dw_1)\,ds\Bigg)\\
&\leq C\int_0^T\mathbb{E}\Bigg(\int_W\int_W\int_W\int_{W\times W}\big(\rho_4^2(w_1)+\rho_4^2(w_2)+\rho_4^2(w_1')+\rho_4^2(w_2')\big)\\
&\qquad\qquad\cdot\Lambda(w_1,w_2,z,dw_1'\otimes dw_2')\,\nu_s^N(dz)\,\nu_s^N(dw_2)\,\nu_s^N(dw_1)\Bigg)\,ds\\
&\leq C\int_0^T\mathbb{E}\Bigg(\int_W\int_W\int_W\big(2\rho_4^2(w_1)+2\rho_4^2(w_2)+\rho_4^2(z)\big)\,\nu_s^N(dz)\,\nu_s^N(dw_2)\,\nu_s^N(dw_1)\Bigg)\,ds\\
&\leq C\int_0^T\mathbb{E}\big(\langle\nu_s^N,\rho_4^2\rangle\big)\,ds.
\end{aligned}$$

The last integral is bounded, uniformly in N, by Proposition 5.1. □

5.2.5 Evolution equation for σ_t^N in $\mathcal{H}_3{}'$

Recall that our goal is to prove convergence of σ_t^N in $D([0,T],\mathcal{H}_1{}')$. Therefore, a necessary previous step is to make sense of (5.8) as an equation in $\mathcal{H}_1{}'$. We will actually need to show something stronger: σ_t^N can be seen as a semimartingale in $\mathcal{H}_3{}'$, whose semimartingale decomposition takes the form suggested by (5.8). We need the following simple result first (for its proof see Proposition 3.4 of Méléard (1998)):

Lemma 5.4. *For every $N>0$ there is a constant $C(N)>0$ such that*

$$\sup_{t\in[0,T]}\mathbb{E}\big(\|\sigma_t^N\|_{\mathcal{H}_4{}'}\big)\leq C(N).$$

Recall that under our assumptions, J_s^N need not be (and in general is not) a bounded operator on $\mathcal{H}_3$, nor on any other $\mathcal{H}_i$, and in fact $J_s^N(\mathcal{H}_i)$ need not even be contained in $\mathcal{H}_i$, so it does not make complete sense to speak of $\big(J_s^N\big)^*$ as the adjoint operator of J_s^N. Nevertheless, for convenience we will abuse notation by writing $\big(J_s^N\big)^*\sigma_s^N$ to denote the linear functional defined by the following mapping:

$$\varphi\in\mathcal{H}_3\longmapsto\big(J_s^N\big)^*\sigma_s^N(\varphi)=\big\langle\sigma_s^N,J_s^N\varphi\big\rangle\in\mathbb{R}.$$

Part of the proof of the following result will consist in showing that $\big(J_s^N\big)^*\sigma_s^N$ is actually in $\mathcal{H}_3{}'$.

Proposition 5.5. *For each $N > 0$, σ_t^N is an ${\mathcal{H}_3}'$-valued semimartingale, and its Doob–Meyer decomposition is given by*

$$\sigma_t^N = \sigma_0^N + \sqrt{N} M_t^N + \int_0^t \left(J_s^N\right)^* \sigma_s^N \, ds, \tag{5.12}$$

where the above is a Bochner integral in ${\mathcal{H}_3}'$.

Proof. By Theorem 5.3 and the embedding ${\mathcal{H}_4}' \hookrightarrow {\mathcal{H}_3}'$, $\sqrt{N} M_t^N$ is an ${\mathcal{H}_3}'$-valued martingale. Thus, by (5.8), the only thing we need to show is that the integral term makes sense as a Bochner integral in ${\mathcal{H}_3}'$. The first step in doing this is to show that $\left(J_s^N\right)^* \sigma_s^N \in {\mathcal{H}_3}'$ for all $s \in [0, T]$. That is, we need to show that there is a $C > 0$ such that

$$\left|\left\langle \sigma_s^N, J_s^N \varphi \right\rangle\right| \le C \left\|\varphi\right\|_{\mathcal{H}_3} \tag{5.13}$$

for all $\varphi \in \mathcal{H}_3$. Observe that by (5.10) and the embedding $\mathcal{H}_3 \hookrightarrow \mathcal{C}_3$, $J_s^N \varphi \in \mathcal{C}_3$ for $\varphi \in \mathcal{H}_3$, and thus

$$\left|\left\langle \sigma_s^N, J_s^N \varphi \right\rangle\right| \le \left\|\sigma_s^N\right\|_{{\mathcal{C}_3}'} \left\|J_s^N \varphi\right\|_{\mathcal{C}_3} \le C \left\|\sigma_s^N\right\|_{{\mathcal{C}_3}'} \left\|\varphi\right\|_{\mathcal{C}_3} \le C \left\|\sigma_s^N\right\|_{{\mathcal{C}_3}'} \left\|\varphi\right\|_{\mathcal{H}_3}$$

for such a function φ by (B1), so (5.13) holds almost surely by Lemma 5.4 and (B1′).

To see that the Bochner integral is (almost surely) well defined, we recall (see Section V.5 in Yosida (1995)) that it is enough to prove that: (i) given any function F in the dual of ${\mathcal{H}_3}'$, the mapping $s \longmapsto F\left(\left(J_s^N\right)^* \sigma_s^N\right)$ is measurable; and (ii) $\int_0^T \left\|\left(J_s^N\right)^* \sigma_s^N\right\|_{{\mathcal{H}_3}'} ds < \infty$. (i) is satisfied by the continuity assumptions on the parameters and (ii) follows from (5.13), using the fact that the constant C there can be chosen uniformly in s. □

We omit the proof of the following corollary (see Corollary 3.8 of Méléard (1998)):

Corollary 5.6. *For any $N > 0$, the process σ_t^N has paths in $D([0, T], {\mathcal{H}_3}')$.*

5.2.6 Uniform estimate for σ_t^N on ${\mathcal{C}_2}'$

Having given sense to equation (5.12) in ${\mathcal{H}_3}'$, we can now give a uniform estimate for σ_t^N in ${\mathcal{C}_2}'$. This will be crucial for obtaining the tightness of σ_t^N in the proof of Theorem 2.

Theorem 5.7.

$$\sup_{N>0} \sup_{t \in [0,T]} \mathbb{E}\left(\left\|\sigma_t^N\right\|_{{\mathcal{C}_2}'}^2\right) < \infty.$$

Proof. By (5.12) and the embedding ${\mathcal{H}_3}' \hookrightarrow {\mathcal{C}_2}'$,

$$\mathbb{E}\left(\left\|\sigma_t^N\right\|_{{\mathcal{C}_2}'}^2\right) \le 2\mathbb{E}\left(\left\|\sigma_0^N\right\|_{{\mathcal{C}_2}'}^2\right) + 2\mathbb{E}\left(\left\|\sqrt{N} M_t^N\right\|_{{\mathcal{C}_2}'}^2\right) + 2\mathbb{E}\left(\left\|\int_0^t \left(J_s^N\right)^* \sigma_s^N \, ds\right\|_{{\mathcal{C}_2}'}^2\right).$$

The first expectation on the right side is bounded uniformly in N by (4.2), and the same holds for the second one by (B1′) and Theorem 5.3. For the last expectation we have

$$\mathbb{E}\left(\left\|\int_0^t \left(J_s^N\right)^* \sigma_s^N\, ds\right\|_{\mathcal{C}_2{}'}^2\right) \leq \mathbb{E}\left(\left[\int_0^t \left\|\left(J_s^N\right)^* \sigma_s^N\right\|_{\mathcal{C}_2{}'} ds\right]^2\right)$$

$$\leq T\int_0^t \mathbb{E}\left(\left\|\left(J_s^N\right)^* \sigma_s^N\right\|_{\mathcal{C}_2{}'}^2\right) ds \leq CT\int_0^T \mathbb{E}\left(\sup_{s\in[0,t]} \left\|\sigma_s^N\right\|_{\mathcal{C}_2{}'}^2\right) dt,$$

where we used Corollary V.5.1 of Yosida (1995) in the first inequality and (5.10) in the last one. Thus by Gronwall's Lemma we get $\mathbb{E}\Big(\sup_{t\in[0,T]} \left\|\sigma_t^N\right\|_{\mathcal{C}_2{}'}^2\Big) \leq C_1 e^{C_2 T}$, uniformly in N, and the result follows. □

5.2.7 Proof of the theorem

We are finally ready to prove Theorem 2.

Proof of Theorem 2. As before, we will proceed in several steps.

Step 1. Our first goal is to show that the sequence of processes σ_t^N is tight in $D([0,T],\mathcal{H}_1{}')$. By Aldous' criterion (which we take from Theorem 2.2.2 in Joffe and Métivier (1986) and the corollary that precedes it in page 34), we need to prove that the following two conditions hold:

(t1) For every rational $t\in[0,T]$ and every $\varepsilon>0$, there is a compact $K\subseteq\mathcal{H}_1{}'$ such that

$$\sup_{N>0} \mathbb{P}\big(\sigma_t^N \notin K\big) \leq \varepsilon.$$

(t2) If $\mathfrak{T}_T^N$ is the collection of stopping times with respect to the natural filtration associated to σ_t^N that are almost surely bounded by T, then for every $\varepsilon>0$,

$$\lim_{r\to 0}\limsup_{N\to\infty} \sup_{\substack{s<r\\ \tau\in\mathfrak{T}_T^N}} \mathbb{P}\big(\left\|\sigma_{\tau+s}^N-\sigma_\tau^N\right\|_{\mathcal{H}_1{}'}>\varepsilon\big)=0.$$

Observe that since the embedding of $\mathcal{H}_2{}'$ into $\mathcal{H}_1{}'$ is compact, (t1) will follow once we show that for any $\varepsilon>0$ and $t\in[0,T]$ there is an $L>0$ such that $\sup_{N>0}\mathbb{P}\Big(\left\|\sigma_t^N\right\|_{\mathcal{H}_2{}'}>L\Big)<\varepsilon$. This follows directly from Markov's inequality, (B1′), and Theorem 5.7, since given any $\varepsilon>0$,

$$\sup_{N>0}\mathbb{P}\Big(\left\|\sigma_t^N\right\|_{\mathcal{H}_2{}'}>L\Big) \leq \frac{1}{L^2}\sup_{N>0}\mathbb{E}\Big(\left\|\sigma_t^N\right\|_{\mathcal{H}_2{}'}^2\Big) \leq \frac{1}{L^2}\sup_{N>0}\mathbb{E}\Big(\left\|\sigma_t^N\right\|_{\mathcal{C}_2{}'}^2\Big) < \varepsilon$$

for large enough L.

To obtain (t2) we will use the semimartingale decomposition of σ_t^N in $\mathcal{H}_3{}'$ given in Proposition 5.5, i.e., $\sigma_t^N=\sigma_0^N+\sqrt{N}M_t^N+\int_0^t\left(J_s^N\right)^*\sigma_s^N\,ds$. By Rebolledo's criterion (see Corollary 2.3.3 in Joffe and Métivier (1986)), (t2) is obtained for the martingale term $\sqrt{N}M_t^N$ if it is proved for the trace of its Doob–Meyer process $\langle\!\langle\sqrt{N}M^N\rangle\!\rangle_t$ in $\mathcal{H}_1$, and thus for σ_t^N if it is proved moreover for the finite variation term $\int_0^t\left(J_s^N\right)^*\sigma_s^N\,ds$ (σ_0^N is tight by hypothesis).

We start with the martingale part. Let τ be a stopping time bounded by T and let $s > 0$. Let $(\phi_k)_{k\geq 0}$ be an orthonormal complete basis of $\mathcal{H}_1$. Using the same calculations as in the proof of Theorem 5.3 we get

$$\begin{aligned}\mathbb{E}\Big(\Big|\mathrm{tr}_{\mathcal{H}_1}\langle\!\langle\sqrt{N}M^N\rangle\!\rangle_{\tau+s} - \mathrm{tr}_{\mathcal{H}_1}\langle\!\langle\sqrt{N}M^N\rangle\!\rangle_{\tau}\Big|\Big)\\ = \mathbb{E}\Big(\int_\tau^{\tau+s}\int_W\int_W\int_W\int_{W\times W}\sum_{k\geq 0}\left(\phi_k(w_1') - \phi_k(w_1) + \phi_k(w_2') - \phi_k(w_2)\right)^2\\ \cdot\Lambda(w_1, w_2, z, dw_1'\otimes dw_2')\,\nu_s^N(dz)\,\nu_s^N(dw_2)\,\nu_s^N(dw_1)\Big)\\ \leq Cs,\end{aligned}$$

uniformly in N. Thus by Markov's inequality,

$$\mathbb{P}\Big(\Big|\mathrm{tr}_{\mathcal{H}_1}\langle\!\langle\sqrt{N}M^N\rangle\!\rangle_t - \mathrm{tr}_{\mathcal{H}_1}\langle\!\langle\sqrt{N}M^N\rangle\!\rangle_t\Big| > \varepsilon\Big) \leq \frac{1}{\varepsilon}Cs,$$

whence (t2) follows for the martingale term.

For the integral term we have that

$$\begin{aligned}\mathbb{E}\bigg(\Big\|\int_0^{\tau+s}\left(J_r^N\right)^*\sigma_r^N\,dr - \int_0^{\tau}\left(J_r^N\right)^*\sigma_r^N\,dr\Big\|_{\mathcal{H}_1{}'}\bigg) \leq \mathbb{E}\Big(\int_\tau^{\tau+s}\Big\|\left(J_r^N\right)^*\sigma_r^N\Big\|_{\mathcal{C}_2{}'}\,dr\Big)\\ \leq C\int_\tau^{\tau+s}\mathbb{E}\Big(\big\|\sigma_r^N\big\|_{\mathcal{C}_2{}'}\Big)\,dr \leq Cs\sup_{r\in[0,T]}\sqrt{\mathbb{E}\Big(\|\sigma_r^N\|_{\mathcal{C}_2{}'}^2\Big)}\end{aligned}$$

for some $C > 0$, uniformly in N, where we used Corollary V.5.1 of Yosida (1995) as before and (B1′) in the first inequality and (5.10) in the second one. Using Markov's inequality as before and Theorem 5.7 we obtain (t2) for the integral term.

Step 2. We have now that every subsequence of σ_t^N has a further subsequence which converges in distribution in $D([0,T],\mathcal{H}_1{}')$. Consider a convergent subsequence of σ_t^N, which we will still denote by σ_t^N, and let σ_t be its limit in $D([0,T],\mathcal{H}_1{}')$. Observe that the only jumps of σ_t^N are those coming from ν_t^N and, with probability 1, at most two agents jump at the same time. Suppose that there is a jump at time t, involving agents i and j. Then given $\varphi\in\mathcal{H}_1$,

$$\begin{aligned}\left|\langle\sigma_t^N,\varphi\rangle - \langle\sigma_{t-}^N,\varphi\rangle\right| &= \frac{1}{\sqrt{N}}\left|\varphi(\eta_t^N(i)) + \varphi(\eta_t^N(j)) - \varphi(\eta_{t-}^N(i)) - \varphi(\eta_{t-}^N(j))\right|\\ &\leq \frac{C}{\sqrt{N}}\|\varphi\|_{\mathcal{H}_1}\left[\sup_{r\in[0,t]}\rho_1(\eta_r^N(i)) + \sup_{r\in[0,t]}\rho_1(\eta_r^N(j))\right]\end{aligned}$$

by (B2). We deduce by (5.7) that

$$\mathbb{E}\left(\sup_{s\in[0,t]}\left\|\sigma_s^N - \sigma_{s-}^N\right\|_{\mathcal{H}_1{}'}^2\right) \leq \frac{C}{N} \tag{5.15}$$

and hence $\sup_{s\in[0,t]}\left\|\sigma_s^N - \sigma_{s-}^N\right\|_{\mathcal{H}_1{}'}$ converges in probability to 0 as $N\to\infty$. Therefore, σ_t is almost surely strongly continuous by Proposition 3.26 of Jacod and Shiryaev (1987). That is, we have shown that every limit point of σ_t^N is (almost surely) in $C([0,T],\mathcal{H}_1{}')$.

Step 3. Our next goal is to prove that the sequence of martingales $\sqrt{N}M_t^N$ converges in distribution in $D([0,T],{\mathcal{H}_1}')$ to the centered Gaussian process Z_t defined in the statement of the theorem. That is, we need to show that given any $\varphi_1,\varphi_2\in\mathcal{H}_1$, the sequence of $\mathbb{R}^2$-valued martingales $\sqrt{N}M_t^{N,(\varphi_1,\varphi_2)}=\left(\sqrt{N}M_t^{N,\varphi_1},\sqrt{N}M_t^{N,\varphi_2}\right)$ converges in distribution to $(Z_t(\varphi_1),Z_t(\varphi_2))$.

By (5.12), $\sqrt{N}M_t^N$ and σ_t^N have the same jumps, and thus (5.15) implies that

$$\mathbb{E}\left(\sup_{s\in[0,t]}\left|\sqrt{N}M_s^{N,(\varphi_1,\varphi_2)}-\sqrt{N}M_{s-}^{N,(\varphi_1,\varphi_2)}\right|^2\right)\xrightarrow[N\to\infty]{}0. \tag{5.16}$$

On the other hand, we claim that for every $\varphi_1,\varphi_2\in\mathcal{H}_1$,

$$\lim_{N\to\infty}\mathbb{E}\Big(\Big\langle\sqrt{N}M^{N,\varphi_1},\sqrt{N}M^{N,\varphi_2}\Big\rangle_t\Big)=\int_0^t C_s^{\varphi_1,\varphi_2}\,ds. \tag{5.17}$$

(5.16) and (5.17) imply that $\sqrt{N}M_t^{N,(\varphi_1,\varphi_2)}$ satisfies the hypotheses of the Martingale Central Limit Theorem (see Theorem VII.1.4 in Ethier and Kurtz (1986)) so, assuming that (5.17) holds, we get that $\sqrt{N}M_t^{N,(\varphi_1,\varphi_2)}$ converges in distribution in $D([0,T],\mathbb{R}^2)$ to $(Z_t(\varphi_1),Z_t(\varphi_2))$.

To prove (5.17) it is enough to consider the case $\varphi_1=\varphi_2=\varphi$, the general case follows by polarization. Given $\mu\in D([0,T],{\mathcal{H}_1}')$ let

$$\begin{aligned}\Psi_t(\mu)=\int_0^t\int_W\int_W\int_W\int_{W\times W}(\varphi(w_1')+\varphi(w_2')-\varphi(w_1)-\varphi(w_2))^2\,\Lambda(w_1,w_2,z,dw_1'\otimes dw_2')\\ \cdot\,\mu_s(dz)\,\mu_s(dw_2)\,\mu_s(dw_1)\,ds.\end{aligned}$$

Then we need to prove that $\lim_{N\to\infty}\mathbb{E}(\Psi_t(\nu^N))=\Psi_t(\nu)$. Let $p>1$ be the exponent we assumed to be such that $\rho_1^p\leq C\rho_4$ for some $C>0$.. Repeating the calculations in the proof of Theorem 5.3 and using Jensen's inequality we get that

$$|\Psi_t(\nu^N)|^p\leq\left[C_1t\,\|\varphi\|_{\mathcal{H}_1}^2\sup_{s\in[0,t]}\left\langle\nu_s^N,\rho_1^2\right\rangle\right]^p\leq C_2t^p\,\|\varphi\|_{\mathcal{H}_1}^{2p}\sup_{s\in[0,t]}\left\langle\nu_s^N,\rho_4^2\right\rangle.$$

Thus Proposition 5.1 implies that the sequence $\left(\Psi_t(\nu^N)\right)_{N>0}$ is uniformly integrable, whence we deduce the desired convergence.

Step 4. As in Step 2, let σ_t be a limit point of σ_t^N. Observe that by the embedding ${\mathcal{H}_1}'\hookrightarrow{\mathcal{C}_0}'$, σ_t^N converges in distribution to σ_t in $D([0,T],{\mathcal{C}_0}')$. We want to prove now that σ_t satisfies (S2-w).

Fix $\varphi\in\mathcal{C}_0$. By (5.12),

$$\begin{aligned}&\langle\sigma_t,\varphi\rangle-\langle\sigma_0,\varphi\rangle-\int_0^t\langle\sigma_s,J_s\varphi\rangle\,ds-Z_t(\varphi)\\ &\quad=\left[\sqrt{N}M_t^{N,\varphi}-Z_t(\varphi)\right]+\left[\langle\sigma_t,\varphi\rangle-\left\langle\sigma_t^N,\varphi\right\rangle\right]+\left[\left\langle\sigma_0^N,\varphi\right\rangle-\langle\sigma_0,\varphi\rangle\right]\\ &\qquad+\int_0^t\left[\left\langle\sigma_s^N,J_s^N\varphi\right\rangle-\left\langle\sigma_s^N,J_s\varphi\right\rangle\right]ds+\int_0^t\left[\left\langle\sigma_s^N,J_s\varphi\right\rangle-\langle\sigma_s,J_s\varphi\rangle\right]ds,\end{aligned} \tag{5.18}$$

so we need to show that the right side converges in distribution to 0 as $N \to \infty$. The first term goes to 0 by the previous step. The next two go to 0 because σ_t is a limit point of σ_t^N and, since $J_s\varphi \in \mathcal{C}_0$, the last term goes to 0 for the same reason.

To show that the remaining term in (5.18) also goes to 0 in distribution, it is enough to show that

$$\mathbb{E}\left(\left|\int_0^t \left\langle \sigma_s^N, \left(J_s^N - J_s\right)\varphi\right\rangle ds\right|\right) \xrightarrow[N\to\infty]{} 0. \tag{5.19}$$

Since, by (5.10), $J_s^N - J_s$ maps $\mathcal{C}_0$ into itself, we get by using (B1$'$) and (5.11) that

$$\begin{aligned}\left|\left\langle \sigma_s^N, \left(J_s^N - J_s\right)\varphi\right\rangle\right| &\leq \left\|\sigma_s^N\right\|_{\mathcal{C}_0{}'} \left\|\left(J_s^N - J_s\right)\varphi\right\|_{\mathcal{C}_0} \leq C\left\|\sigma_s^N\right\|_{\mathcal{C}_2{}'} \|\varphi\|_{\mathcal{C}_0} \left\|\nu_s^N - \nu_s\right\|_{\mathcal{C}_2{}'} \\ &= \frac{C}{\sqrt{N}} \|\varphi\|_{\mathcal{C}_0} \left\|\sigma_s^N\right\|_{\mathcal{C}_2{}'}^2 .\end{aligned}$$

(5.19) now follows from this bound and Theorem 5.7.

Step 5. We have shown in Step 4 that if σ_t is any accumulation point of σ_t^N, then σ_t satisfies (S2-w) for every $\varphi \in \mathcal{C}_0$. To see that the limit points of σ_t^N actually solve (S2), the only thing left to show is that the integral term in (S2) makes sense as a Bochner integral in $\mathcal{C}_0{}'$. This can be verified by repeating the arguments of the proof of Proposition 5.5.

Step 6. We want to prove now pathwise uniqueness for the solutions of (S2). Fix a centered Gaussian process Z_t in $\mathcal{C}_0{}'$ with the right covariance structure and suppose that $\sigma_t^1, \sigma_t^2 \in \mathcal{C}_0$ are two solutions of (S2) for this choice of Z_t. Then $\sigma_t^1 - \sigma_t^2 = \int_0^t \left(J_s^*\sigma_s^1 - J_s^*\sigma_s^2\right) ds$, so

$$\sup_{t\in[0,T]} \left\|\sigma_t^1 - \sigma_t^2\right\|_{\mathcal{C}_0{}'} \leq \int_0^T \sup_{s\in[0,t]} \left\|J_s^*\left(\sigma_s^1 - \sigma_s^2\right)\right\|_{\mathcal{C}_0{}'} dt.$$

By (5.10), J_s is a bounded operator on $\mathcal{C}_0$, and thus so is J_s^* as an operator on $\mathcal{C}_0{}'$. Moreover, $\|J_s^*\|_{\mathcal{C}_0{}'}$ can be bounded uniformly in s. Thus

$$\mathbb{E}\left(\sup_{t\in[0,T]} \left\|\sigma_t^1 - \sigma_t^2\right\|_{\mathcal{C}_0{}'}\right) \leq C\int_0^t \mathbb{E}\left(\sup_{s\in[0,t]} \|\sigma_s^1 - \sigma_s^2\|_{\mathcal{C}_0{}'}\right) dt,$$

and Gronwall's Lemma implies that $\sigma_t^1 = \sigma_t^2$ for all $t \in [0,T]$ almost surely, so the pathwise uniqueness for (S2) follows.

Step 7. We have now that any accumulation point σ_t of the sequence σ_t^N satisfies equation (S2), which has a unique pathwise solution. The last thing left to show is the uniqueness in law for the solutions of this equation. Since we have pathwise uniqueness, this can be obtained by adapting the Yamada–Watanabe Theorem to our setting (see Theorem IX.1.7 of Revuz and Yor (1999)). The proof works in the same way assuming we can construct regular conditional probabilities in $D([0,T], \mathcal{C}_0{}')$, which is possible in any complete metric space (see Theorem I.4.12 of Durrett (1996)). This (together with the embedding $\mathcal{H}_1{}' \hookrightarrow \mathcal{C}_0{}'$) implies that (S2) determines a unique process in $C([0,T], \mathcal{H}_1{}')$. □

5.3 Proof of Theorems 3 and 4a-4d

Proof of Theorem 3. There are three conditions to check. The first one, $\sigma_0^N \Longrightarrow \sigma_0$ in ${\mathcal{H}_1}'$, follows directly from applying the Central Limit Theorem in $\mathbb{R}$ to each of the processes $\langle \sigma_0^N, \varphi \rangle$ for $\varphi \in \mathcal{H}_1$, while the condition $\sup_{N>0} \mathbb{E}\big(\langle \nu_0^N, \rho_4^2 \rangle\big) < \infty$ is straightforward. For the remaining one we can prove something stronger, namely that $\sup_{N>0} \mathbb{E}\Big(\left\|\sigma_0^N\right\|_{\mathcal{H}_4'}^2\Big) < \infty$. In fact, if $(\phi_k)_{k\geq 0}$ is a complete orthonormal basis of $\mathcal{H}_4$ and η_0^N is chosen by picking the type $\eta_0^N(i)$ of each agent $i \in I_N$ independently according to ν_0 then

$$\mathbb{E}\Big(\left\|\sigma_0^N\right\|_{\mathcal{H}_4'}^2\Big) = \mathbb{E}\bigg(\sum_{k\geq 0} \langle \sigma_0^N, \phi_k \rangle^2\bigg) = \frac{1}{N} \sum_{k\geq 0} \mathbb{E}\bigg(\Big[\sum_{i=1}^N \big[\phi_k(\eta_0^N(i)) - \langle \nu_0, \phi_k \rangle\big]\Big]^2\bigg).$$

A simple computation and (B2) (see the proof of Proposition 3.5 in Méléard (1998)) show that this is bounded by $\mathbb{E}\big(\langle \nu_0^N, \rho_4^2 \rangle\big) + \langle \nu_0, \rho_4^2 \rangle$, which is in turn bounded by some $C < \infty$ uniformly in N, so the result follows. □

For Theorems 4a (finite W), 4c ($W = \Omega \subseteq \mathbb{R}^d$ smooth and compact), and 4d ($W = \mathbb{R}^d$), we already explained why the assumptions of Theorem 2 hold, so the results follow directly from that theorem (together with (4.3) when W is finite). We are left with the case $W = \mathbb{Z}^d$.

Proof of Theorem 4b. Let $\varphi \in \ell^\infty(\mathbb{Z}^d)$. Then

$$\|\varphi\|_{2,D}^2 = \sum_{x\in\mathbb{Z}^d} \frac{\varphi(x)^2}{1+|x|^{2D}} \leq C \|\varphi\|_\infty^2,$$

where we used the fact that $2D > d$ implies that $\sum_{x\in\mathbb{Z}^d}(1+|x|^{2D})^{-1} < \infty$. This gives the embedding $\ell^\infty(\mathbb{Z}^d) \hookrightarrow \ell^{2,D}(\mathbb{Z}^d)$. The other continuous embeddings in (4.4) are similar. To see that the embedding $\ell^{2,D}(\mathbb{Z}^d) \hookrightarrow \ell^{2,2D}$ is compact, observe that the family $(e_y)_{y\in\mathbb{Z}^d} \subseteq \ell^{2,D}(\mathbb{Z}^d)$ defined by $e_y(x) = \sqrt{1+|x|^{2D}}\mathbf{1}_{x=y}$ defines an orthonormal complete basis of $\ell^{2,D}(\mathbb{Z}^d)$ and, using the same fact as above,

$$\sum_{y\in\mathbb{Z}^d} \|e_y\|_{2,2D}^2 = \sum_{y\in\mathbb{Z}^d} \frac{1+|y|^{2D}}{1+|y|^{4D}} < \infty,$$

so the embedding is Hilbert–Schmidt, and hence compact. (B2) and (B3) follow directly from the definition of the spaces in this case.

(D1) and (D2) for ρ_4^2 are precisely what is assumed in Theorem 4b, and using this and Jensen's inequality we get the same estimates for ρ_1^2, ρ_2^2, and ρ_3^2. We are left checking (D3). For simplicity we will assume here that $\Lambda \equiv 0$. For (D3.i), the case $\mathcal{C}_0 = \ell^\infty(\mathbb{Z}^d)$ is straightforward. Now if $\left\langle \mu_i, 1+|\cdot|^{8D}\right\rangle < \infty$, $i=1,2$, and $\varphi \in \ell^{\infty,2D}(\mathbb{Z}^d)$,

$$\begin{aligned}
\left|\frac{J_{\mu_1,\mu_2}\varphi(z)}{1+|z|^{2D}}\right| &= \frac{1}{1+|z|^{2D}} \sum_{x\in\mathbb{Z}^d}\sum_{y\in\mathbb{Z}^d} \big(\varphi(y)-\varphi(x)\big)\Gamma(x,z,\{y\})\mu_1(\{x\}) \\
&\qquad + \frac{1}{1+|z|^{2D}} \sum_{x\in\mathbb{Z}^d}\sum_{y\in\mathbb{Z}^d} \big(\varphi(y)-\varphi(z)\big)\Gamma(z,x,\{y\})\mu_2(\{x\}) \\
&\le \frac{C\,\|\varphi\|_{\infty,2D}}{1+|z|^{2D}}\Bigg[\sum_{x\in\mathbb{Z}^d}\sum_{y\in\mathbb{Z}^d}\left(1+|y|^{2D}\right)\,\Gamma(x,z,\{y\})\big(\mu_1(\{x\})+\mu_2(\{x\})\big) \\
&\qquad + \sum_{x\in\mathbb{Z}^d}\left(1+|x|^{2D}\right)\mu_1(\{x\}) + 1 + |z|^{2D}\Bigg] \\
&\le \frac{C\,\|\varphi\|_{\infty,2d}}{1+|z|^{2D}}\left[1+|z|^{2D} + \sum_{x\in\mathbb{Z}^d}\left(1+|x|^{2D}\right)\mu_2(\{x\})\right] \le C\,\|\varphi\|_{\infty,2D}
\end{aligned}$$

uniformly in z, where we used (4.5a) with a power of $2D$ instead of $8D$. We deduce that $\|J_{\mu_1,\mu_2}\varphi\|_{\infty,2D} \le C\,\|\varphi\|_{\infty,2D}$ as required. The proof for $\ell^{\infty,3D}(\mathbb{Z}^d)$ is similar. For (D3.ii), consider $\varphi\in\ell^{\infty}(\mathbb{Z}^d)$ and $\mu_1,\mu_2,\mu_3,\mu_4\in\mathcal{P}$. Then

$$\begin{aligned}
|(J_{\mu_1,\mu_2}-J_{\mu_3,\mu_4})\varphi(z)| &= \left|\int_W \Gamma\varphi(w;z)\big[\mu_1(dw)-\mu_3(dw)\big] + \int_W \Gamma\varphi(z;w)\big[\mu_2(dw)-\mu_4(dw)\big]\right| \\
&\le \|\Gamma\varphi(\cdot;z)\|_\infty\,\|\mu_1-\mu_3\|_{\ell^\infty(\mathbb{Z}^d)'} + \|\Gamma\varphi(z;\cdot)\|_\infty\,\|\mu_2-\mu_4\|_{\ell^\infty(\mathbb{Z}^d)'}\,.
\end{aligned}$$

Now $\|\Gamma\varphi(\cdot;z)\|_\infty$ and $\|\Gamma\varphi(z;\cdot)\|_\infty$ are both bounded by $4\overline{\lambda}\,\|\varphi\|_\infty$, so we get

$$\|(J_{\mu_1,\mu_2}-J_{\mu_3,\mu_4})\varphi\|_\infty \le 4\overline{\lambda}\,\|\varphi\|_\infty\left[\|\mu_1-\mu_3\|_{\ell^{\infty,2d}(\mathbb{Z}^d)'} + \|\mu_2-\mu_4\|_{\ell^{\infty,2d}(\mathbb{Z}^d)'}\right]$$

as required. □

Acknowledgements. I am grateful to the three members of my PhD thesis committee for their help during the course of this research. In particular, I thank Philip Protter for suggesting to me the problem which originated this paper, Laurent Saloff-Coste for helpful discussions about the functional analytical setting used in Theorem 2, and my advisor, Rick Durrett, for his invaluable help throughout my research and for his suggestions and comments on several versions of the manuscript. I also want to thank an anonymous referee for a very careful reading of the manuscript and for helpful comments and suggestions.

References

Adams, R. A. (2003). *Sobolev Spaces*. Academic Press, second edition.

Bayraktar, E., Horst, U., and Sircar, S. (2007). Queueing theoretic approaches to financial price fluctuations. *arXiv:math/0703832*.

Billingsley, P. (1999). *Convergence of probability measures*. Wiley Series in Probability and Statistics: Probability and Statistics. John Wiley & Sons Inc., New York, second edition.

Blackwell, D. and Dubins, L. E. (1983). An extension of Skorohod's almost sure representation theorem. *Proc. Amer. Math. Soc.*, **89**(4), 691–692.

Dai Pra, P., Runggaldier, W. J., Sartori, E., and Tolotti, M. (2007). Large portfolio losses; a dynamic contagion model. *arXiv:0704.1348v1 [math.PR]*.

Davis, M. H. A. and Esparragoza-Rodriguez, J. C. (2007). Large portfolio credit risk modeling. *Int. J. Theor. Appl. Finance*, **10**(4), 653–678.

Duffie, D. and Manso, G. (2007). Information percolation in large markets. *American Economic Review, Papers and Proceedings*, **97**, 203–209.

Duffie, D. and Sun, Y. (2007). Existence of independent random matching. *Ann. Appl. Probab.*, **17**(1), 386–419.

Duffie, D., Gârleanu, N., and Pedersen, L. H. (2005). Over-the-counter markets. *Econometrica*, **73**(6), 1815–1847.

Durrett, R. (1996). *Probability: theory and examples*. Duxbury Press, Belmont, CA, second edition.

Ethier, S. N. and Kurtz, T. G. (1986). *Markov processes: characterization and convergence*. Wiley Series in Probability and Mathematical Statistics. John Wiley & Sons Inc., New York.

Ferrari, P. A. and Marić, N. (2007). Quasi stationary distributions and Fleming-Viot processes in countable spaces. *Electron. J. Probab.*, **12**, no. 24, 684–702 (electronic).

Fleming, W. H. and Viot, M. (1979). Some measure-valued Markov processes in population genetics theory. *Indiana Univ. Math. J.*, **28**(5), 817–843.

Föllmer, H. (1974). Random economies with many interacting agents. *J. Math. Econom.*, **1**(1), 51–62.

Fournier, N. and Méléard, S. (2004). A microscopic probabilistic description of a locally regulated population and macroscopic approximations. *Ann. Appl. Probab.*, **14**(4), 1880–1919.

Giesecke, K. and Weber, S. (2004). Cyclical correlations, credit contagion, and portfolio losses. *Journal of Banking and Finance*, **28**(12), 3009–3036.

Huck, S. and Kosfeld, M. (2007). The dynamics of neighbourhood watch and norm enforcement. *The Economic Journal*, **117**(516), 270–286.

Jacod, J. and Shiryaev, A. N. (1987). *Limit theorems for stochastic processes*, volume 288 of *Grundlehren der Mathematischen Wissenschaften [Fundamental Principles of Mathematical Sciences]*. Springer-Verlag, Berlin.

Joffe, A. and Métivier, M. (1986). Weak convergence of sequences of semimartingales with applications to multitype branching processes. *Adv. in Appl. Probab.*, **18**(1), 20–65.

Kufner, A. (1980). *Weighted Sobolev Spaces*. Leipzig.

Méléard, S. (1998). Convergence of the fluctuations for interacting diffusions with jumps associated with Boltzmann equations. *Stochastics Stochastics Rep.*, **63**(3-4), 195–225.

Métivier, M. (1987). Weak convergence of measure valued processes using Sobolev-imbedding techniques. In *Stochastic partial differential equations and applications (Trento, 1985)*, volume 1236 of *Lecture Notes in Math.*, pages 172–183. Springer, Berlin.

Revuz, D. and Yor, M. (1999). *Continuous martingales and Brownian motion*, volume 293 of *Grundlehren der Mathematischen Wissenschaften [Fundamental Principles of Mathematical Sciences]*. Springer-Verlag, Berlin, third edition.

Yosida, K. (1995). *Functional analysis*. Classics in Mathematics. Springer-Verlag, Berlin. Reprint of the sixth (1980) edition.