Polynuclear growth and the KPZ fixed point

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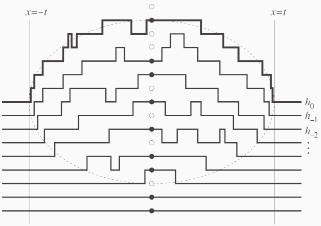
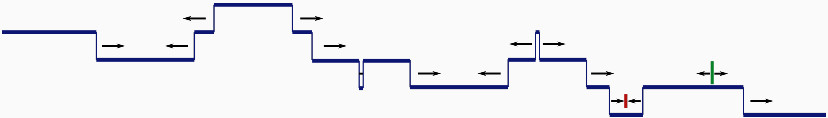


Joint work with Konstantin Matetski and Jeremy Quastel

The polynuclear growth process (PNG)



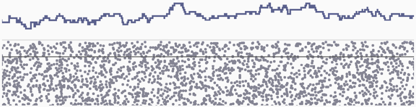
Height function $h: \mathbb{R} \longrightarrow \mathbb{Z} \cup \{-\infty\}$ [Gates-Westcott '95, Prähofer-Spohn '02]

Down jumps move to the right, up jumps move to the left, at speed 1

Up/down jump nucleations at space-time Poisson pt. proc., rate 2

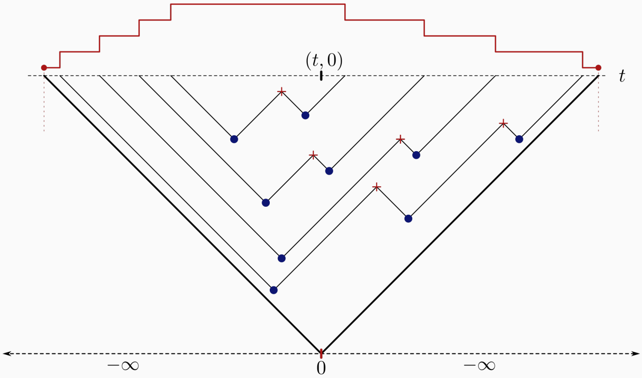
Expanding islands annihilate on contact

Independent Poisson up and down jumps, rate 1, are invariant



[Simulation by Patrik Ferrari]

Droplet/narrow wedge initial condition



Blue dots are creation events | Red crosses are annihilations

Initial condition is $\mathfrak{d}_0(0) = 0$, $\mathfrak{d}_0(x) = -\infty$ if $x \neq 0$ narrow wedge

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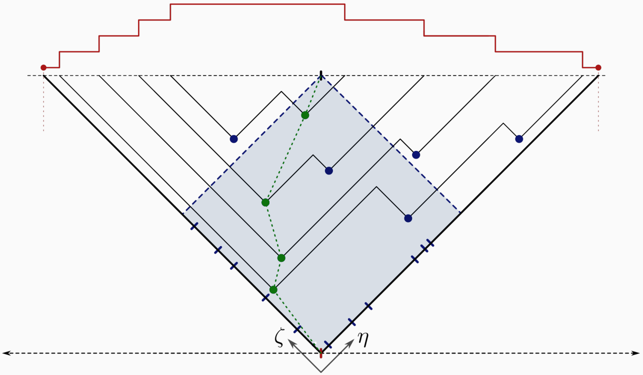

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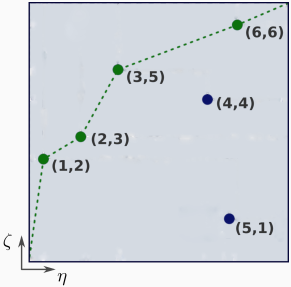
Initial condition is $\mathfrak{d}_0(0) = 0,\ \mathfrak{d}_0(x) = -\infty$ if $x \neq 0$ narrow wedge

$$h(t, x; h_0) = \sup_{y \in \mathbb{R}} \left\{ h(t, x; \mathfrak{d}_y) + h_0(y) \right\} \qquad \text{(Poissonian LPP)}$$

Droplet/narrow wedge initial condition



$N \sim \text{Poisson}[t^2]$ points inside the square, η-ζ order defines $\sigma \in S_N$

$$\sigma = (2, 3, 5, 4, 6, 1)$$

$h(t, 0)$ is the length ℓ_N of longest increasing subsequence in σ

[Baik-Deift-Johansson '99] $\quad N^{-1/6}(\ell_N - 2\sqrt{N}) \longrightarrow$ Tracy-Widom GUE

So $\quad t^{-1/3}(h(t, 0) - 2t) \longrightarrow$ Tracy-Widom GUE

Droplet/narrow wedge initial condition

Gaussian Unitary Ensemble

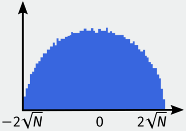
An $N \times N$ GUE matrix is a random Hermitian matrix

$$A = \begin{bmatrix} A_{1,1} & A_{1,2} & \cdots & A_{1,N} \\ * & A_{2,2} & \cdots & A_{1,N} \\ & & \ddots & \vdots \\ * & * & \ddots & \vdots \\ * & * & \cdots & A_{N,N} \end{bmatrix} \quad \text{with} \quad \begin{cases} A_{i,j} = \mathcal{N}(0, \tfrac{1}{2}) + \mathrm{i}\mathcal{N}(0, \tfrac{1}{2}) & \text{if } i > j, \\ A_{i,i} = \mathcal{N}(0, 2). \end{cases}$$

The *largest eigenvalue* $\lambda_{\mathrm{GUE}}(N)$ satisfies [Tracy-Widom '94]

$$N^{1/6}\big(\lambda_{\mathrm{GUE}}(N) - 2\sqrt{N}\big) \xrightarrow[N\to\infty]{(d)} \zeta_{\mathrm{GUE}}$$

(Tracy-Widom GUE distribution)



N

h

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$t^{-1/3}(h(t,0) - 2t) \longrightarrow$ Tracy-Widom GUE

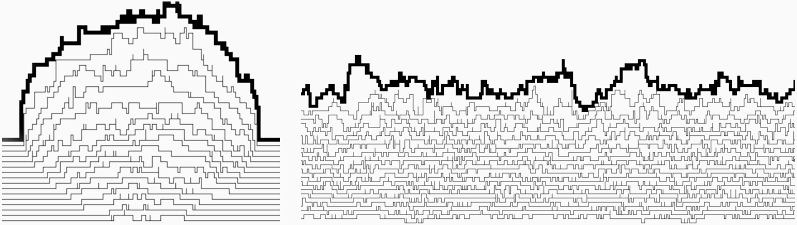
Narrow wedge/droplet: $t^{-1/3}(h(t, t^{2/3}x) - 2t) \longrightarrow \mathscr{A}_2(x) - x^2$

[Prähofer-Spohn '02, Johansson '03] Airy$_2$ process, TW-GUE marginals

Flat ($h_0 \equiv 0$): $t^{-1/3}(h(t, t^{2/3}x) - 2t) \longrightarrow \mathscr{A}_1(x)$

[Borodin-Ferrari-Sasamoto '08] Airy$_1$ process, TW-GOE marginals

(symmetrized random permutations [Baik-Rains '01])



Similar results for exclusion processes, LPP ('00s)

Many extensions for related processes during the last decade

KPZ universality

PNG belongs to the KPZ universality class

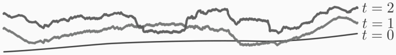
(random interface growth, directed polymers, interacting particle systems...)

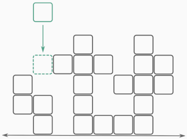
$\rightsquigarrow$ one expects that (assuming $\varepsilon^{1/2} h_0(c_2 \varepsilon^{-1} x) \longrightarrow \mathfrak{h}_0(x)$ as $\varepsilon \to 0$)

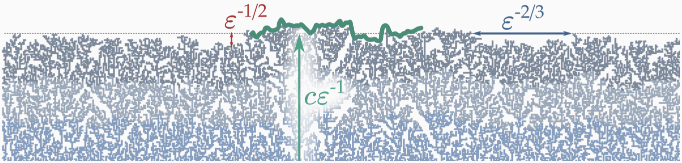
$$\varepsilon^{1/2} h(c_1 \varepsilon^{-3/2} t, c_2 \varepsilon^{-1} x) - c_3 \varepsilon^{-1} t \xrightarrow[\varepsilon \to 0]{} \mathfrak{h}(t, x; \mathfrak{h}_0)$$

where $\big(\mathfrak{h}(t,x)\big)_{t \geq 0, x \in \mathbb{R}}$ is the KPZ fixed point (starting from $\mathfrak{h}_0$)

The KPZ fixed point is a 1:2:3 scaling invariant Markov process which is the (conjectural/defining) universal scaling limit of all models in the KPZ class







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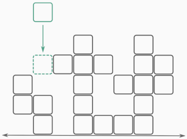
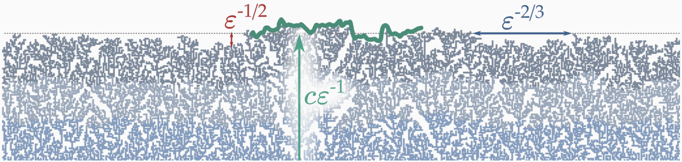
Constructed as a scaling limit of TASEP in [Matetski-Quastel-R '17], known now for several other special models





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Thm:

The above limit holds for the PNG model

Follows from [Dauvergne-Virág ’21] or [Matetski-Quastel-R ’22]

[Dauvergne-Virág ’21]: $\sup_{y \in \mathbb{Z}} \big(h(t, x; \mathfrak{d}_y) + h_0(y)\big) \longrightarrow \sup_{y \in \mathbb{Z}} \big(\mathscr{A}_t(x, y) + \mathfrak{h}_0(y)\big)$ (directed landscape)

Fredholm determinant formula for PNG

[Matetski-Quastel-R '22]: explicit Fredholm determinant formula for the PNG transition probs. $\mathbb{P}_{h_0}\big(h(t,x_i) \le r_i,\ i = 1,\dots,n\big)$

<u>One-point distributions:</u> $\mathbb{P}_{h_0}\big(h(t,x) \le r\big) = \det\big(I - K^{h_0}_{t,x}\big)_{\ell^2(\mathbb{Z}_{>r})}$

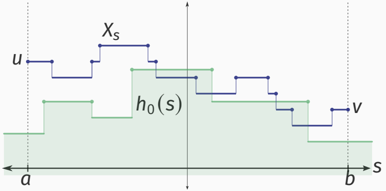
with $K^{h_0}_{t,x} = e^{-2t\nabla - t\Delta}\, P^{\mathrm{hit}(h_0)}_{x-t,x+t}\, e^{2t\nabla - t\Delta}$

(multipoint formulas are similar)

$$\nabla f(x) = \tfrac{1}{2}\big(f(x+1) - f(x-1)\big), \qquad \Delta f(x) = f(x+1) - 2f(x) + f(x-1),$$

X_s a symmetric, continuous time, rate 1, nearest neighbor RW on $\mathbb{Z}$

$$P^{\mathrm{hit}(h_0)}_{a,b}(u,v) = \mathbb{P}_{X_a = u}\big(X \text{ hits hypo}(h_0) \text{ on } [a,b],\ X_b = v\big)$$



Sketch of the proof: Kolmogorov equation

The formula was first obtained as the scaling limit of a similar formula for discrete time TASEP with parallel update [Matetski-R '21]

Alternative pf.: check directly that $F_r(t,x)$ satisfies the *Kolmogorov backw. eqn.*

Let $\mathscr{L}$ be the generator of PNG, $\mathscr{L} = \mathscr{L}_{\mathrm{det}} + \mathscr{L}_{\mathrm{cr}}$

Need to check that $\boxed{\partial_t F_r(t,x;h) = \mathscr{L} F_r(t,x;h)}$ ($\mathscr{L}$ acting on initial h)

$F_r(t,x;h) = \det(I-K)$, so $\partial_t F_r(t,x;h) = -\det(I-K)\,\mathrm{tr}((I-K)^{-1}\partial_t K)$

$\mathscr{L}_{\mathrm{det}}$ is a differ. oper., $\mathscr{L}_{\mathrm{cr}} \det(I-K)$ is built out of rank one perturbations,

so $$\mathscr{L} F_r(t,x;h) = -\det(I-K)\,\mathrm{tr}((I-K)^{-1}\mathscr{L}K)$$

Thus we only need to prove $\boxed{\partial_t K = \mathscr{L} K}$ (!)

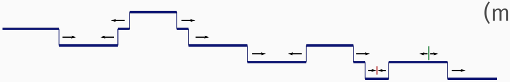
$\partial_t K$ is explicit: $\boxed{\partial_t K = \partial_t\left(e^{-2t\nabla - t\Delta} P^{\mathrm{hit}(h_0)}_{x-t,x+t} e^{2t\nabla - t\Delta}\right) = -2[\nabla, K]}$

$[\nabla, K] = \nabla K - K\nabla$

So we need $\mathscr{L}K = -2[\nabla, K]$.

Key fact 1: $g(s) := X_s$ is invariant under the PNG dynamics

(modulo global height)



Key fact 2: the PNG dynamics preserves order,

$$f_1 \le f_2 \iff h(t,\cdot;f_1) \le h(t,\cdot;f_2)$$

Then

$$\begin{aligned}
\mathscr{L}_h P^{\mathrm{hit}(h)}_{a,b}(u,v) &= \mathscr{L}_h \mathbb{E}_{a,u;b,v}(\Phi^{\mathrm{hit}\,0}(g-h)) \qquad (g(s)=X_s)\\
&= \lim_{t\to 0} \tfrac{1}{t}\Big(\mathbb{E}_{a,u;b,v}(\Phi^{\mathrm{hit}\,0}(g-h(t))) - \mathbb{E}_{a,u;b,v}(\Phi^{\mathrm{hit}\,0}(g-h))\Big)\\
&= \lim_{t\to 0} \tfrac{1}{t}\Big(\mathbb{E}_{a,u;b,v}(\Phi^{\mathrm{hit}\,0}(g(-t)-h)) - \mathbb{E}_{a,u;b,v}(\Phi^{\mathrm{hit}\,0}(g-h))\Big)\\
&= \mathbb{E}_{a,u;b,v}\Big(\mathscr{L}^*_g \Phi^{\mathrm{hit}\,0}(g-h)\Big)\\
&= \cdots\cdots\\
&= -2[\nabla, P^{\mathrm{hit}(h)}_{a,b}]
\end{aligned}$$

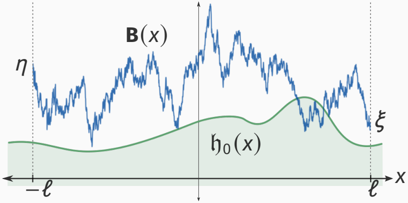
KPZ fixed point formula

$$K^{h_0}_{t,x} = e^{-2t\nabla - t\Delta} P^{\text{hit}(h_0)}_{x-t,x+t} e^{2t\nabla - t\Delta}.$$

After ε rescaling: $X_s \longrightarrow B_s$,

$e^{x\Delta} \longrightarrow e^{x\partial^2}$ (heat kernel)

$e^{2t\nabla} \approx e^{\frac{t}{3}\partial^3 + \text{shift}}$



$\mathbf{K}^{\text{hypo}(\mathfrak{h}_0)}_{\text{Brownian}} \sim$ asymptotic probab. for a BM to hit hypograph$(\mathfrak{h}_0)$

Brownian scattering operator

For the KPZ fixed point,

$$\mathbb{P}_{\mathfrak{h}_0}(\mathfrak{h}(t,x) \le r) = \det\left(\mathbf{I} - e^{-\frac{t}{3}\partial^3 - x\partial^2} \mathbf{K}^{\text{hypo}(\mathfrak{h}_0)}_{\text{Brownian}} e^{\frac{t}{3}\partial^3 + x\partial^2}\right)_{L^2((r,\infty))}$$

(Similar formula for multipoint distributions)

Classical integrability of PNG/KPZ fixed point

[Baik-Deift-Johansson '99]: for PNG with narrow wedge initial data,

$$F_r(s) := \mathbb{P}_{\mathfrak{d}_0}(h(s,0) \leq r) = \det\big(I_{i-j}(2s)\big)_{i,j=0,\ldots,r-1}$$

(I_k modified Bessel functions of the first kind)

From this, $\alpha_r^2 = 1 - \frac{F_{r+1}F_{r-1}}{F_r^2}$ satisfies [Periwal-Shevitz '90, Borodin '01 ...]

$$-s(1-\alpha_r^2)(\alpha_{r+1}+\alpha_{r-1}) = r\alpha_r$$ discrete Painlevé II equation

as well as $\partial_s \alpha_r = \frac{1}{2}(1-\alpha_r^2)(\alpha_{r+1}-\alpha_{r-1})$ Ablowitz-Ladik lattice

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For the limiting fluctuations one has $F_{\text{TW-GUE}}(x) = e^{-\int_r^\infty (x-z)q(z)^2 dz}$

with q the *Hastings-McLeod* solution of [Tracy-Widom '94]

$$q''(x) = 2q(x)^3 + xq(x)$$ Painlevé II equation

Similar results for flat initial data

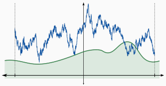
Stochastic integrability of PNG/TASEP/KPZ fixed point

$$\partial_t K^{\mathrm{PNG}} = -2\big[\nabla, K^{\mathrm{PNG}}\big], \qquad \partial_t K^{\text{fixed pt}} = -\tfrac{1}{3}\big[\partial^3, K^{\text{fixed pt}}\big]$$

$$\partial_x K^{\mathrm{PNG}} = -2\nabla K^{\mathrm{PNG}} - 2K^{\mathrm{PNG}}\nabla, \qquad \partial_x K^{\text{fixed pt}} = -\tfrac{1}{2}\partial^2 K^{\text{fixed pt}} - \tfrac{1}{2}K^{\text{fixed pt}}\partial^2$$

$\mathbf{K}^{h_0}$ $\longleftrightarrow$ (linear evol.) $\big(\mathbf{K}_t^{h_0}\big)_{t>0}$

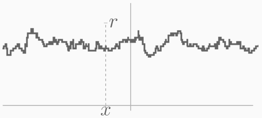
$\mathbf{K}^{h_0}$ $\longleftrightarrow$ h_0

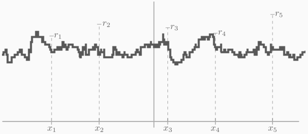


$\big(\mathbf{K}_t^{h_0}\big)_{t>0}$ $\longrightarrow$ (Fredholm det.) $\mathbb{P}_{h_0}\big(h(t,x_i) \le r_i,\ i = 1,\dots,n\big)$

h_0 $\longrightarrow$ (transition probabs.) $\mathbb{P}_{h_0}\big(h(t,x_i) \le r_i,\ i = 1,\dots,n\big)$

Integrable equations for PNG





Thm: [Matetski-Quastel-R '22]

$F_r(t,x) = \mathbb{P}(h(t,x) \leq r)$

satisfies the 2D Toda equation

$$\tfrac{1}{4}(\partial_t^2 - \partial_x^2)\log F_r = \frac{F_{r+1}F_{r-1}}{F_r^2} - 1$$

$F_r(t,x) = 0$ for $r > \sup_{|y-x|\leq t} h_0(y)$

for any UC PNG initial condition h_0.

Multipoint version:

$F(t,\vec{x},\vec{r}) = \mathbb{P}\big(h(t,x_i) \leq r_i,\ i = 1,\dots,n\big)$

$x = x_1 + \cdots + x_n,\ r = (r_1,\dots,r_n)$

$$\frac{F(t,x_1,\dots,x_n,r_1+1,\dots,r_n+1)}{F(t,x_1,\dots,x_n,r_1,\dots,r_n)} = \det Q_r$$

with Q_r satisfying the non-Abelian 2D Toda equations

$$\tfrac{1}{4}\partial_{t-x}(\partial_{t+x}Q_rQ_r^{-1}) + Q_rQ_{r-1}^{-1} - Q_{r+1}Q_r^{-1} = 0.$$

Flat: $g_r(t) = \log F_r(2t) - \log F_{r-1}(2t)$ satisfies $\ddot{g}_r = e^{g_{r+1}-g_r} - e^{g_r - g_{r-1}}$

(classic Toda lattice)

KPZ 1:2:3 scaling limits

$$\varepsilon^{1/2} h(\varepsilon^{-3/2}t, \varepsilon^{-1}x) - 2\varepsilon^{-1}t \quad \text{(PNG)} \quad \xrightarrow[\varepsilon \to 0]{} \quad \mathfrak{h}(t,x) \quad \text{(KPZ fixed pt.)}$$

$$\rightsquigarrow \quad \text{under } F(t,x,r) \longmapsto F(\varepsilon^{3/2}t, \varepsilon x, \varepsilon^{1/2}(x-2t)),$$

$$\tfrac{1}{4}(\partial_t^2 - \partial_x^2) \log F_r = \frac{F_{r+1}F_{r-1}}{F_r^2} - 1 \quad \text{(2D Toda)}$$

$$\xrightarrow[\varepsilon \to 0]{\phi = \partial_r^2 \log F} \quad \partial_r\left(\partial_t \phi + \phi\, \partial_r \phi + \tfrac{1}{12}\partial_r^3 \phi\right) + \tfrac{1}{4}\partial_x^2 \phi = 0$$

(KP equation)

And there's a multipoint version: $\partial_r \log(F) = \operatorname{tr}(\mathbf{Q})$ with the $n \times n$ matrices $\mathbf{Q}$ and $\mathbf{q} = \partial_r \mathbf{Q}$ solving

$$\partial_t \mathbf{q} + \tfrac{1}{2}\partial_r \mathbf{q}^2 + \tfrac{1}{12}\partial_r^3 \mathbf{q} + \tfrac{1}{4}\partial_x^2 \mathbf{Q} + \tfrac{1}{2}(\mathbf{q}\,\partial_x \mathbf{Q} - \partial_x \mathbf{Q}\mathbf{q}) = 0.$$

(matrix KP equation)

Limiting equations derived earlier in [Quastel-R '19]

From KP to Tracy-Widom

The Tracy-Widom distributions arise simply from the 1:2:3 scaling invariance of the KPZ fixed point as self-similar solutions of KP

For $\mathfrak{h}_0$ = flat: $\phi(t,x,r) = \partial_r^2 \log(F(t,x,r))$ does not depend on x,

it solves $\partial_t\phi + \frac{1}{12}\partial_r^3\phi + \frac{1}{2}(\partial_r\phi)^2 = 0$ (KdV equation)

Write $\phi(t,x,r) = 4^{2/3}t^{-2/3}\psi(4^{1/3}t^{-1/3}r)$ with $\psi = \frac{1}{2}(q' - q^2)$

(Miura's transform)

$\Longrightarrow q'' = 2q^3 + rq$ Painlevé II equation

$\Longrightarrow \mathbf{F}(1,0,r) = e^{-\frac{1}{2}\int_r^\infty (q(s)+(r-s)q(s)^2)\,ds}$ $\rightsquigarrow$ recovers TW-GOE

Similar for narrow-wedge/TW-GUE