Exact formulas for random growth off a flat interface

Daniel Remenik

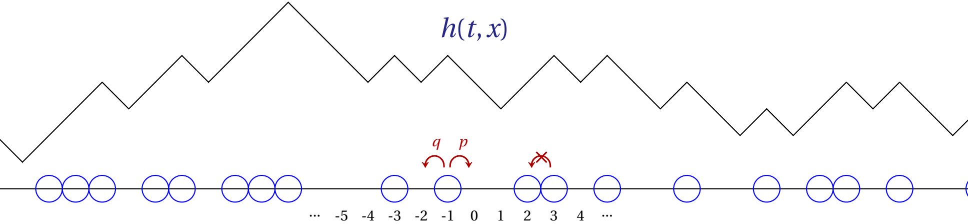
Joint with Janosch Ortmann and Jeremy Quastel

Universidad de Chile

EBP, August 2014

Asymmetric Simple Exclusion Process (ASEP)

local min $\mapsto$ local max at rate q
local max $\mapsto$ local min at rate p



$\eta_t(x) \in \{0,1\}$ is the occupation variable.

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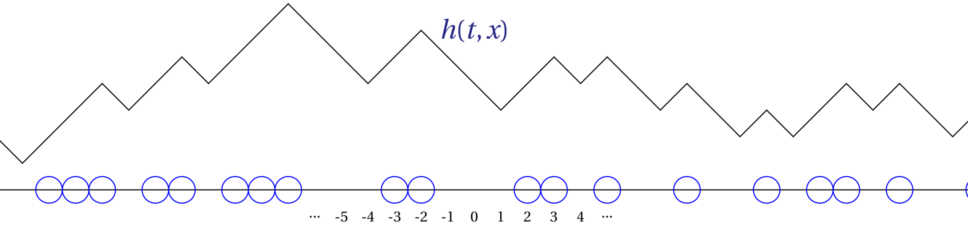

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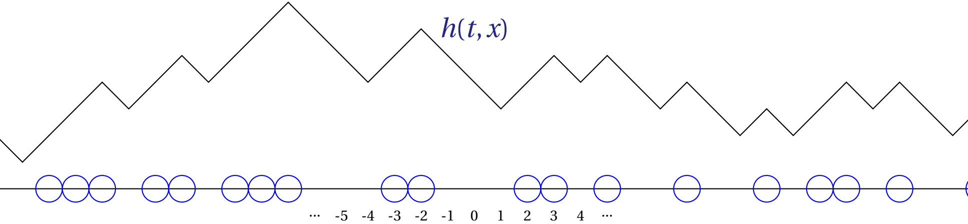


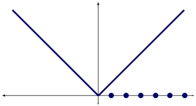
$\eta_t(x) \in \{0,1\}$ is the occupation variable.

We will take $q \geq p$, so the interface tends to grow.

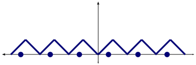
The case $p = 0$, $q = 1$ (TASEP) is equivalent to last passage percolation.

Three special classes of initial data (scale invariance)

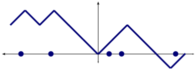
- Step / curved: $\eta_0(x) = \mathbf{1}_{x\geq 0} \quad \rightsquigarrow \quad h(0,x) = |x|.$



- Flat / periodic: $\eta_0(x) = \mathbf{1}_{x\in 2\mathbb{Z}} \quad \rightsquigarrow \quad h(0,x) = \frac{1}{2}(1+(-1)^x).$



- Stationary / Bernoulli: $\eta_0 =$ product of Bernoullis
  $\rightsquigarrow \quad h(0,x) =$ SRW path



There are also three "mixed" cases.

Weakly asymmetric limit

One reason for interest in ASEP is that it has an adjustable asymmetry $\gamma = q - p$.

Setting $\gamma = \varepsilon^{1/2}$ one has [Bertini-Giacomin '98, Amir-Corwin-Quastel '10]

$$\varepsilon^{1/2} h(\varepsilon^{-2} t, \varepsilon^{-1} x) - C_\varepsilon(t,x) \xrightarrow[\varepsilon \to 0]{} H(t,x)$$

where $H(t,x)$ solves the Kardar-Parisi-Zhang (KPZ) equation

$$\partial_t H = \tfrac{1}{2}\partial_x^2 H + \tfrac{1}{2}(\partial_x H)^2 + \xi$$

with ξ space-time white noise.

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Hence the ASEP height function can be thought of as a discretization of KPZ.

TASEP ($q = 1 - p = 1$) asymptotics

For TASEP/LPP these six cases can be fully analyzed. Exact computations based on determinantal structure give exact limiting distributions.

[2000-07: Baik, Deift, Johansson, Rains, Borodin, Ferrari, Prähofer, Spohn, Sasamoto,...]

For curved i.c., $h(0,x) = |x|$, we have

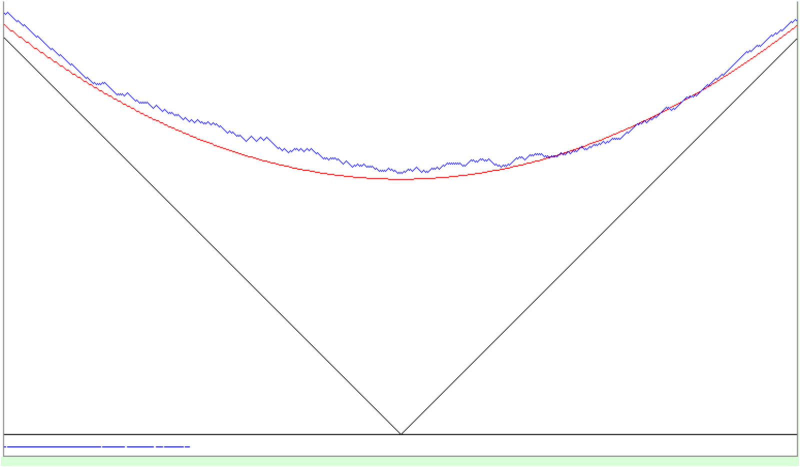
$$\mathbb{P}^{\text{TASEP}}\left(\frac{h(t,0) - \frac{1}{2}t}{t^{1/3}} \geq -r\right) \xrightarrow[t\to\infty]{} F_{\text{GUE}}(2^{1/3}r).$$

For flat i.c., $h(0,x) = \frac{1}{2}(1 + (-1)^x)$, we have

$$\mathbb{P}^{\text{TASEP}}\left(\frac{h(t,x) - \frac{1}{2}t}{t^{1/3}} \geq -r\right) \xrightarrow[t\to\infty]{} F_{\text{GOE}}(2r).$$

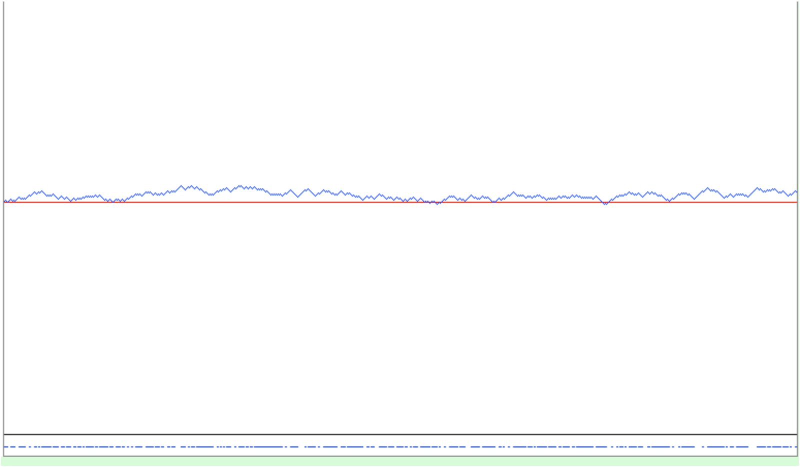
Limits given in terms of Fredholm determinants are natural in view of the determinantal structure.

There are also multipoint results, leading to the Airy processes.



Difference between the blue and red curves is an approximate Airy$_2$ process. At each x the distribution is F_{GUE}.

(Simulation by Patrik Ferrari)



Difference between the blue and red curves is an approximate Airy$_1$ process. At each x the distribution is F_{GOE}.

(Simulation by Patrik Ferrari)

Exact formulas for step ASEP

$q > p \quad \rightsquigarrow$ non-determinantal/"positive temperature" model

By *KPZ universality* one expects the same scaling and limiting distributions in the partially asymmetric (ASEP) case.

Most well understood case is the step initial condition:

[Tracy-Widom '08-'09]

$$\mathbb{P}^{\text{step}}\big(h(t/(q-p),x) \geq y\big) = \frac{1}{2\pi \mathrm{i}} \oint \frac{d\mu}{\mu} \prod_{k=0}^{\infty} \big(1-\mu\big(\tfrac{p}{q}\big)^k\big) \det(I-J),$$

$$J(\eta,\eta') = \oint d\zeta \, \frac{e^{\Psi(\zeta)-\Psi(\eta')}}{\eta'(\zeta-\eta)} \sum_{k=-\infty}^{\infty} \frac{\big(\frac{p}{q}\big)^k \big(\frac{\zeta}{\eta'}\big)^k}{\mu - \big(\frac{p}{q}\big)^k}, \qquad \Psi(\zeta) = -x\log(1-\zeta) + \frac{t\zeta}{(1-\zeta)} + (y+x)\log\zeta.$$

Obtained by Bethe ansatz. Combinatorial miracles (Cauchy determinant, Ramanujan summation formula, etc...) lead to the Fredholm determinant.

Significant "post-processing" yields a formula suitable for asymptotic analysis and the conjectured GUE asymptotics [Tracy-Widom '08].

The same method yields the conjectured asymptotics for the step-Bernoulli case [Tracy-Widom '09].

For step and step-Bernoulli, [Amir-Corwin-Quastel '10, C-Q '11] worked out the weakly asymmetric (i.e. KPZ) limit of these formulas.

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The flat cases remain unclear.

- Exact formulas for the ASEP distribution function [Lee '10]. Not suitable for asymptotics, not even at a formal level.
- Non-rigorous physics derivations for flat and half-flat KPZ [Le Doussal-Calabrese '12, Le Doussal '14].

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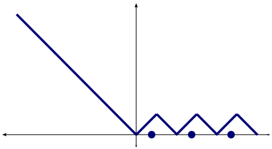
Basic idea of physics derivation:

- Let $Z(t,x) = e^{H(t,x)}$ with $\partial_t H = \frac{1}{2}(\partial_x H)^2 + \frac{1}{2}\partial_x^2 H + \xi$ (KPZ eq.).
- Find $\mathbb{E}[Z(t,x)^n]$ by Bethe ansatz.
- Leads to formulas for $\mathbb{E}[Z(t,x)^n]$. Sum the moments and interchange limits to write $\mathbb{E}[e^{-\zeta Z}]$ as $\sum_{n=0}^{\infty} \frac{(-\zeta)^n}{n!}\mathbb{E}[Z^n]$.
- But $\mathbb{E}[Z^n] \approx e^{\frac{1}{24}n(n^2-1)}$! Moment problem is ill-posed.

Our results: [Ortmann-Quastel-R '14]

- Exact formulas for half-flat and flat ASEP.
- Fredholm Pfaffian in the flat case.
- Formal critical point analysis leads to the conjectured asymptotics. Rigorous proofs are work in progress.

I will focus first on the half-flat case $\eta_0(x) = \mathbf{1}_{x \in 2\mathbb{Z}_{>0}}$.



ASEP duality [Sasamoto-Imamura '10, Borodin-Corwin-Sasamoto '13]

Suppose there is a *left-most particle*. Let $N_x(t)$ be the net number of particles which have crossed to from the left to the right of site x up to time t. Define

$$\tau = \frac{p}{q}.$$

Then

$$u(t;\vec{x}) := \mathbb{E}\left[\tau^{N_{x_1-1}(t)}\eta_{x_1}(t)\cdots\tau^{N_{x_k-1}(t)}\eta_{x_k}(t)\right]$$

is the unique solution of

(1) $\partial_t u(t,\vec{x}) = \sum_{a=1}^{k}\left[pu(t,\vec{x}_a^-) + qu(t,\vec{x}_a^+) - u(t,\vec{x})\right]$

$\vec{x}_a^{\pm} = (x_1,\dots,x_a \pm 1,\dots,x_k)$

(2) When $x_1 \le \ldots \le x_k$, $x_{a+1} = x_a + 1$,

$$pu(t,\vec{x}_{a+1}^-) + qu(t,\vec{x}_a^+) = u(t,\vec{x}).$$

(3) For $x_1 < x_2 < \cdots < x_k$

$$u(0,\vec{x}) = u_0(\vec{x}).$$

- Borodin and Corwin found contour integral formulas for moments of the q-Whittaker process, arising from their study of Macdonald processes.
- Suitable scaling limit yields formulas for the moments of the delta Bose gas with δ_0 initial condition:

  (1)' $\partial_t v(t,\vec{x}) = \frac{1}{2}\Delta v(t,\vec{x})$,

  (2)' $\left[\frac{\partial}{\partial_{x_a}} - \frac{\partial}{\partial_{x_{a+1}}} - 1\right] v(t,\vec{x}) = 0$ when $x_1 \leq \cdots \leq x_k$ and $x_a = x_{a+1}$,

  (3)' $k! \int_{x_1 < \cdots < x_k} d\vec{x}\, v(0,\vec{x}) f(\vec{x}) = f(0)$ for suitable f.

  Here $v(t,\vec{x})$ can be interpreted as $v(t,\vec{x}) = \mathbb{E}\left[\prod_a e^{H(t,x_a)}\right]$.
- By analogy, this allowed them to find contour integral formulas for ASEP with step (and step-Bernoulli) initial conditions:

$$u(t,\vec{x}) = \frac{1}{(2\pi \mathrm{i})^k} \int_{C^k} d\vec{z} \prod_{1\leq a<b\leq k} \frac{z_a - z_b}{z_a - \tau z_b} \prod_{a=1}^{k} \frac{\left(\frac{1-\tau z_a}{1-z_a}\right)^{x_a - 1} e^{t(p\frac{1-z_a}{1-\tau z_a} + q\frac{1-\tau z_a}{1-z_a} - 1)}}{1 - z_a}$$

C is a small circle around 1

ASEP with half-flat initial condition translates into the initial condition

$$u_0(\vec{x}) = \tau^{-k} \prod_{a=1}^{k} 1_{x_a \in 2\mathbb{Z}_{>0}} \tau^{\frac{1}{2} x_a}.$$

Our solution of the equations in this case is based on a careful study of E. Lee's half-flat ASEP formula

$$\mathbb{P}\big(N_x(t) \geq m\big) = (-1)^m \sum_{k \geq m} \frac{\tau^{(k-m)(k-m+1)/2}}{(1+\tau)^{k(k-1)} k!} \begin{bmatrix} k-1 \\ k-m \end{bmatrix}_\tau$$
$$\times \int_{C_R^k} d\vec{\xi} \; \prod_i \frac{\xi_i^x e^{t(p\xi_i^{-1} + q\xi_i - 1)}}{(1-\xi_i)(\xi_i^2 - \tau)}$$
$$\times \prod_{i \neq j} \frac{\xi_j - \xi_i}{p + q\xi_i\xi_j - \xi_i} \prod_{i<j} \frac{1 + \tau - (\xi_i + \xi_j)}{\tau - \xi_i\xi_j}.$$

Contour integral ansatz for half-flat ASEP

(1) $\partial_t u(t,\vec{x}) = \sum_{a=1}^{k} \left[p u(t,\vec{x}_a^-) + q u(t,\vec{x}_a^+) - u(t,\vec{x}) \right]$,

(2) $p u(t,\vec{x}_{a+1}^-) + q u(t,\vec{x}_a^+) - u(t,\vec{x}) = 0$ when $x_{a+1} = x_a + 1$,

(3) $u(0,\vec{x}) = \tau^{-k} \prod_{a=1}^{k} 1_{x_a \in 2\mathbb{Z}_{>0}} \tau^{\frac{1}{2} x_a}$ for $x_1 < x_2 < \cdots < x_k$,

is solved by

$$u(t,\vec{x}) = \frac{\tau^{\frac{1}{2}k(k-1)}}{(2\pi \mathrm{i})^k} \int_{C^k} d\vec{z} \prod_{1 \le a < b \le k} \frac{z_a - z_b}{z_a - \tau z_b} \frac{1 - z_a z_b}{1 - \tau z_a z_b} \times \prod_{a=1}^{k} \frac{\left(\frac{1-\tau z_a}{1-z_a}\right)^{x_a - 1} e^{t\left(p \frac{1-z_a}{1-\tau z_a} + q \frac{1-\tau z_a}{1-z_a} - 1\right)}}{\tau z_a^2 - 1}.$$

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C is a small circle around 1

In the case of half-flat q-TASEP and half-flat semi-discrete polymers, it appears that such simple formulas do not exist.

Half-flat KPZ moments

We can similarly solve the delta Bose gas with initial condition $v(0,\vec{x}) = \prod_a e^{-\theta x}\mathbf{1}_{x\geq 0}$:

$$v(t,\vec{x}) := \mathbb{E}\left[\prod_{i=1}^{k} Z(t,x_i)\right] = \int_{\vec{\delta}+(\mathrm{i}\mathbb{R})^k} \frac{d\vec{z}}{(2\pi \mathrm{i})^k} \prod_{a<b}\left(\frac{z_a - z_b}{z_a - z_b - 1}\,\frac{z_a + z_b - 1}{z_a + z_b}\right) \times \prod_{a=1}^{k} \frac{1}{z_a} e^{\frac{t}{2}\sum_{a=1}^{k}(z_a-\theta)^2 + \sum_{a=1}^{k}(z_a-\theta)x_a}$$

where $\delta_1 > \delta_2 + 1 > \cdots > \delta_k + k - 1 > k - 1$ and $x_1 < \cdots < x_k$.

After some post-processing we recover Le Doussal-Calabrese's formulas:

$$\mathbb{E}\left[e^{-\zeta Z(t,x)}\right] = \sum_{k=0}^{\infty}\frac{1}{k!}\int_{(\mathrm{i}\mathbb{R})^k} d\vec{u} \sum_{n_1,\dots,n_k=1}^{\infty} \prod_a \frac{2^{n_a}}{n_a}\,\frac{\Gamma(1-2u_a-2\gamma-n_a)}{\Gamma(1-2u_a-2\gamma)}\, e^{(n_a^3-n_a)\frac{t}{12} + n_a t u_a^2 - \gamma n_a s + x n_a u_a} \times \prod_{a<b} \frac{\Gamma(1-u_a-u_b-2\gamma-\frac{n_a+n_b}{2})\Gamma(1-u_a-u_b-2\gamma+\frac{n_a+n_b}{2})}{\Gamma(1-u_a-u_b-2\gamma+\frac{n_a-n_b}{2})\Gamma(1-u_a-u_b-2\gamma-\frac{n_a-n_b}{2})}\,\frac{(n_a-n_b)^2-4(u_a-u_b)^2}{(n_a+n_b)^2-4(u_a-u_b)^2}$$

Half-flat ASEP moments

Recall that we obtained a formula for

$$u(t;\vec{x}) := \mathbb{E}\left[\tau^{N_{x_1-1}(t)}\eta_{x_1}(t)\cdots\tau^{N_{x_k-1}(t)}\eta_{x_k}(t)\right] \qquad \text{with} \qquad x_1 < \cdots < x_k.$$

Additional combinatorics + contour deformation yields

(Imamura-Sasamoto '11, Borodin-Corwin-Sasamoto '13)

$$\mathbb{E}\left[\tau^{mN_x(t)}\right] = m_\tau! \sum_{k=0}^{m} \frac{1}{k!} \sum_{\substack{n_1,\dots,n_k\geq 1\\ n_1+\cdots+n_k=m}} \int_{C^k} \frac{d\vec{w}}{(2\pi\mathrm{i})^k} \det\left[\frac{-1}{w_a\tau^{n_a}-w_b}\right]_{a,b=1}^{k} \times \prod_a f(w_a;n_a)g(w_a;n_a) \prod_{a<b} \alpha(w_a,w_b;n_a,n_b),$$

C contour containing $-1,0$ with $-\tau^{-1},\tau^{-1}$ on outside

with

$$f(w;n) = (1-\tau)^{n_a} e^{t\left[\frac{1}{1+w_a}-\frac{1}{1+\tau^{n_a}w_a}\right]} \left(\frac{1+\tau^{n_a}w_a}{1+w_a}\right)^{x-1},$$

$$g(w;n) = \frac{(-w;\tau)_\infty}{(-\tau^n w;\tau)_\infty}\frac{(\tau^{2n}w^2;\tau)_\infty}{(\tau^n w^2;\tau)_\infty}, \quad \alpha(w_1,w_2;n_1,n_2) = \frac{(w_1w_2;\tau)_\infty(\tau^{n_1+n_2}w_1w_2;\tau)_\infty}{(\tau^{n_1}w_1w_2;\tau)_\infty(\tau^{n_2}w_1w_2;\tau)_\infty}.$$

Here the q-Pochhammer symbol is $\quad (a;q)_\infty = \prod_{\ell=0}^{\infty}(1-q^\ell a).$

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$$\underbrace{\mathbb{E}\left[\tau^{mN_x(t)}\right]}_{\leq 1} = m_\tau! \sum_{k=0}^{m} \frac{1}{k!} \sum_{\substack{n_1,\ldots,n_k\geq 1\\ n_1+\cdots+n_k=m}} \int_{C^k} \frac{d\vec{w}}{(2\pi \mathrm{i})^k} \det\left[\frac{-1}{w_a\tau^{n_a}-w_b}\right]_{a,b=1}^{k}$$

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C contour containing $-1,0$ with $-\tau^{-1},\tau^{-1}$ on outside

with

$$f(w;n) = (1-\tau)^{n_a} e^{t\left[\frac{1}{1+w_a} - \frac{1}{1+\tau^{n_a}w_a}\right]} \left(\frac{1+\tau^{n_a}w_a}{1+w_a}\right)^{x-1},$$

$$g(w;n) = \frac{(-w;\tau)_\infty}{(-\tau^n w;\tau)_\infty}\frac{(\tau^{2n}w^2;\tau)_\infty}{(\tau^n w^2;\tau)_\infty}, \quad \alpha(w_1,w_2;n_1,n_2) = \frac{(w_1w_2;\tau)_\infty(\tau^{n_1+n_2}w_1w_2;\tau)_\infty}{(\tau^{n_1}w_1w_2;\tau)_\infty(\tau^{n_2}w_1w_2;\tau)_\infty}.$$

Here the q-Pochhammer symbol is $\quad (a;q)_\infty = \prod_{\ell=0}^{\infty}(1-q^\ell a).$

τ-Laplace transform and formal asymptotics

Standard q-exponential fn.: $e_q(x) = \dfrac{1}{((1-q)x;q)_\infty} = \sum_{k=0}^{\infty} \dfrac{x^k}{k_q!}$.

(series only valid for $|x| < 1$)

Here $k_q! = [1]_q \cdot [2]_q \cdots [k]_q$, with $[k]_q = \frac{1-q^k}{1-q}$.

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Here $\quad k_q! = [1]_q \cdot [2]_q \cdots [k]_q, \quad$ with $\quad [k]_q = \frac{1-q^k}{1-q}.$

Summing the moment formula + analytic continuation yields

$$\mathbb{E}^{\mathrm{hf}}\left[e_\tau\left(\zeta \tau^{N_x(t)}\right)\right] = \sum_{k=0}^{\infty} \frac{1}{k!} \int_{(\delta+\mathrm{i}\mathbb{R})^k} \frac{d\vec{s}}{(2\pi\mathrm{i})^k} \int_{C_k^k} \frac{d\vec{w}}{(2\pi\mathrm{i})^k} \det\left[\frac{-1}{w_a \tau^{s_a} - w_b}\right]_{a,b=1}^{k}$$
$$\times \prod_a \frac{\pi}{\sin(-\pi s_a)} \zeta^{s_a} f(w_a; s_a) g(w_a; s_a) \prod_{a<b} \alpha(w_a, w_b; s_a, s_b),$$

Here we are rewriting the sums in the n_a's through the Mellin-Barnes integral representation:

$$\sum_{n\geq 1} \zeta^n f(n) = \frac{1}{2\pi\mathrm{i}} \int_{\delta+\mathrm{i}\mathbb{R}} ds \frac{\pi}{\sin(-\pi s)} (-\zeta)^s f(s).$$

This transform determines the distribution of $N_x(t)$.

τ-Laplace transform and formal asymptotics

$$\mathbb{E}^{\mathrm{hf}}\left[e_\tau\left(\zeta\tau^{N_x(t)}\right)\right] = \sum_{k=0}^{\infty}\frac{1}{k!}\int_{(\delta+\mathrm{i}\mathbb{R})^k}\frac{d\vec{s}}{(2\pi\mathrm{i})^k}\int_{C_k^k}\frac{d\vec{w}}{(2\pi\mathrm{i})^k}\det\left[\frac{-1}{w_a\tau^{s_a}-w_b}\right]_{a,b=1}^{k}$$
$$\times\prod_a\frac{\pi}{\sin(-\pi s_a)}\zeta^{s_a}f(w_a;s_a)g(w_a;s_a)\prod_{a<b}\alpha(w_a,w_b;s_a,s_b),$$

To compute (formally) the asymptotics the idea is to use the basic trick that for a fixed sequence $\lambda_m \nearrow \infty$ and a sequence of random variables X_m,

$$\lim_{m\to\infty}\mathbb{E}\left[\exp\left(-e^{-\lambda_m X_m}\right)\right] = \lim_{m\to\infty}\mathbb{P}\left(X_m > 0\right).$$

$e_\tau(x)$ behaves sufficiently like $\exp(x)$ for $x \in (-\infty, 0)$ so that this is still true.

τ-Laplace transform and formal asymptotics

$$\mathbb{E}^{\mathrm{hf}}\big[e_\tau\big(\zeta\tau^{N_x(t)}\big)\big] = \sum_{k=0}^{\infty}\frac{1}{k!}\int_{(\delta+\mathrm{i}\mathbb{R})^k}\frac{d\vec{s}}{(2\pi\mathrm{i})^k}\int_{C_k^k}\frac{d\vec{w}}{(2\pi\mathrm{i})^k}\,\det\Big[\frac{-1}{w_a\tau^{s_a}-w_b}\Big]_{a,b=1}^{k}$$
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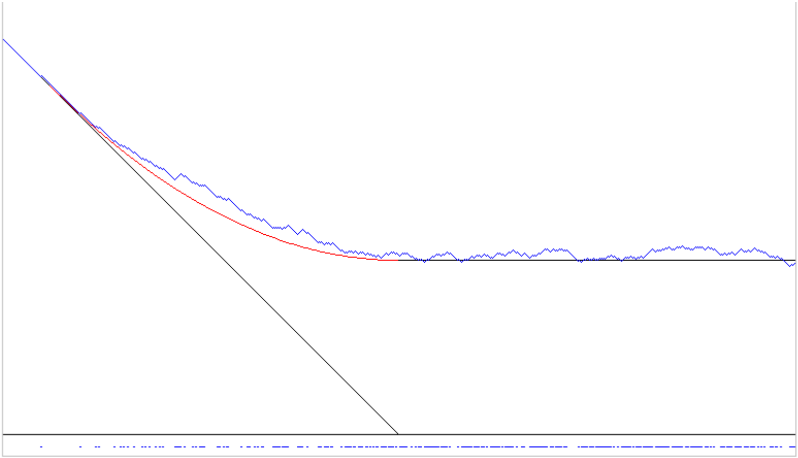
Formal critical point analysis shows that the asymptotic fluctuations are the conjectured ones:

$$\lim_{t\to\infty}\mathbb{P}^{\mathrm{hf}}\left(\frac{h\big(\frac{t}{q-p},t^{2/3}x\big)-\frac{1}{2}t-\frac{1}{2}t^{1/3}x^2\mathbf{1}_{x\geq 0}}{t^{1/3}}>-r\right)$$
$$=\mathbb{P}\big(\mathscr{A}_{2\to 1}(2^{-1/3}x)\leq 2^{1/3}r\big).$$

Unfortunately, $\alpha(w_a,w_b;s_a,s_b)$ is not controlled away from the critical point.



Difference between the blue and red curves is an approximate Airy$_{2\to 1}$ process. The distribution at each x interpolates between F_{GOE} and F_{GUE}.

(Simulation by Patrik Ferrari)

From half-flat to flat

Recall we have

$$\mathbb{E}^{\mathrm{hf}}\big[\tau^{mN_x(t)}\big] = m_\tau! \sum_{k=0}^{m} \frac{1}{k!} \sum_{\substack{n_1,\dots,n_k\geq 1\\ n_1+\cdots+n_k=m}} \int_{C^k} \frac{d\vec{w}}{(2\pi \mathrm{i})^k} \det\Big[\frac{-1}{w_a\tau^{n_a}-w_b}\Big]_{a,b=1}^{k}$$

$$\times \prod_a \left(\frac{1-\tau z_a}{1-z_a}\right)^{x-1} \tilde{f}(w_a;n_a)g(w_a;n_a) \prod_{a<b} \alpha(w_a,w_b;n_a,n_b),$$

From half-flat to flat

Recall we have

$$\mathbb{E}^{\mathrm{hf}}\Big[\tau^{m(N_x(t)-\frac{1}{2}x)}\Big] = m_\tau! \sum_{k=0}^{m} \frac{1}{k!} \sum_{\substack{n_1,\dots,n_k\geq 1\\ n_1+\cdots+n_k=m}} \int_{C^k} \frac{d\vec{w}}{(2\pi \mathrm{i})^k} \det\Big[\frac{-1}{w_a\tau^{n_a}-w_b}\Big]_{a,b=1}^{k}$$

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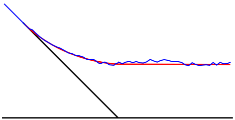

We want to probe into the flat region of the half-wedge: since here the ASEP height function satisfies $h(t,x)=2N_x(t)-x$, we have

$$\mathbb{E}^{\mathrm{flat}}\Big[\tau^{\frac{1}{2}mh(t,0)}\Big] = \lim_{x\to\infty} \mathbb{E}^{\mathrm{hf}}\Big[\tau^{m(N_{2x}(t)-x)}\Big].$$

It turns out that $\left|\frac{1-\tau z_a}{1-z_a}\tau^{-\frac{1}{2}n_a}\right| < 1 \iff |z_a| > \tau^{1-\frac{1}{2}n_a}$.

Thus we have to deform z_a to a circle of radius $R_a > \tau^{1-\frac{1}{2}n_a}$.

Flat formula

- Flat formula is a monster sum of residues collected as we deform each contour to a circle of radius R_a
- Miracle 1: All residues lose dependence on x.

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- There are two types of poles:
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  2. $\prod_a g(w_a; n_a)$ gives rise to *“unpaired variables”*

Flat formula

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- There are two types of poles:
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  2. $\prod_a g(w_a; n_a)$ gives rise to *"unpaired variables"*

We end up with a formula reading like

$$\mathbb{E}^{\text{flat}}(\tau^{\frac{1}{2}mh(t,0)}) = m!_\tau \sum_{k=0}^{m} \sum_{\substack{k_\text{u},k_\text{p}\geq 0\\ k_\text{u}+2k_\text{p}=k}} \frac{1}{k_\text{u}!2^{k_\text{p}}k_\text{p}!} \sum_{(\vec{\varsigma},\vec{n}^\text{u},\vec{n}^\text{p})\in\Lambda^m_{k_\text{u},k_\text{p}}} \times \frac{1}{(2\pi \text{i})^{k_\text{p}}} \int_{C^{k_\text{p}}} d\vec{z}^\text{p}\, I(\vec{n}^\text{u}, \vec{n}^\text{p}; \vec{z}^\text{u}, \vec{z}^\text{p}, \vec{z}^{-\text{p}})$$

with C a circle of radius slightly > 1, $\vec{z}^\text{u}_a = \varsigma_a$ and $z^{-\text{p}}_a = 1/z^\text{p}_a$.

- Miracle 2: Massive cancellations and pairing structure of the variables leads to products of the form

$$\prod_{1\le a<b\le 2k} \frac{y_b - y_a}{y_a y_b - 1} = \mathrm{Pf}\left[\frac{y_b - y_a}{y_a y_b - 1}\right]_{a,b=1}^{2k}$$

(Schur Pfaffian identity)

and

$$\prod_{1\le a<b\le k} (-\varsigma_a \varsigma_b)^{m_a\wedge m_b+1} \mathrm{sgn}(\varsigma_a m_a - \varsigma_b m_b)$$

$$= \mathrm{Pf}\left[(-\varsigma_a\varsigma_b)^{m_a\wedge m_b} \mathrm{sgn}(\varsigma_b m_b - \varsigma_a m_a)\right]_{a,b=1}^{k}$$

for positive integers $m_1, \ldots, m_k$, and $\varsigma_1, \ldots \varsigma_k \in \{-1, 1\}$.

Here

$$\mathrm{Pf}(A) = \frac{1}{2^n n!} \sum_{\sigma\in S_{2n}} \mathrm{sgn}(\sigma) \prod_{i=1}^{n} A_{\sigma(2i-1),\sigma(2i)}, \quad \mathrm{Pf}(A)^2 = \det(A).$$

(A antisymmetric matrix of even size)

- Miracle 3: The result can (almost) be written as a single Pfaffian.

Flat ASEP moment formula:

$$\mathbb{E}^{\mathrm{flat}}\left[\tau^{\frac{1}{2}mh(t,0)}\right] = m!_\tau \sum_{k=0}^{m} \frac{(-1)^{\frac{1}{2}k(k+1)}}{2^k k!} \sum_{\substack{m_1,\dots,m_k=1,\\ m_1+\cdots+m_k=m}}^{\infty} \int_{C_{0,1}^k} d\vec{y}$$

$$\times \prod_{1\le a<b\le k} \tau^{-\frac{1}{2}m_a m_b} \prod_{a=1}^{k} u(y_a, m_a)\,\mathrm{Pf}\left[K(\vec{y};\vec{m})\right]$$

with

$$K_{1,1}(y_a, y_b; m_a, m_b) = 4u_{\mathrm{p}}(y_a, m_a)\mathbf{1}_{m_a=m_b}\delta_{y_a-\frac{1}{y_b}} + (-y_a y_b)^{m_a\wedge m_b}\mathrm{sgn}(y_b m_b - y_a m_a)$$
$$\times\, u_{\mathrm{u}}(y_a, m_a)\,u_{\mathrm{u}}(y_b, m_b)(\delta_{y_a-1}+\delta_{y_a+1})(\delta_{y_b-1}+\delta_{y_b+1}),$$
$$K_{1,2}(y_a, y_b; m_a, m_b) = u_{\mathrm{u}}(-1, m_a)\delta_{y_a+1} - u_{\mathrm{u}}(1, m_a)\delta_{y_a-1},$$
$$K_{2,2}(y_a, y_b; m_a, m_b) = \frac{\tau^{\frac{1}{2}m_a}y_a - \tau^{\frac{1}{2}m_b}y_b}{\tau^{\frac{1}{2}(m_a+m_b)}y_a y_b - 1},$$

$$u(z,n) = \exp\left[t\left(\frac{1}{1+\tau^{-n/2}z} - \frac{1}{1+\tau^{-n/2}z}\right)\right]\tau^n(1-\tau)^n \frac{\left(-\tau^{-n/2}z;\tau\right)_\infty (\tau^{1+n}z^2;\tau)_\infty}{\left(-\tau^{n/2}z;\tau\right)_\infty (\tau z^2;\tau)_\infty},$$
$$u_{\mathrm{u}}(z,n) = \tau^{-\frac{1}{2}n} z \frac{1-\tau^n z^2}{1-\tau^n}, \qquad u_{\mathrm{p}}(z,n) = (-1)^n \tau^{-n}\frac{1+z^2}{z^2-1}.$$

Flat ASEP moment formula:

$$\mathbb{E}^{\text{flat}}\left[\tau^{\frac{1}{2}mh(t,0)}\right] = m!_\tau \sum_{k=0}^{m} \frac{(-1)^{\frac{1}{2}k(k+1)}}{2^k k!} \sum_{\substack{m_1,\ldots,m_k=1,\\ m_1+\cdots+m_k=m}}^{\infty} \int_{C_{0,1}^k} d\vec{y}$$

$$\times \prod_{1\le a<b\le k} \tau^{-\frac{1}{2}m_a m_b} \prod_{a=1}^{k} u(y_a, m_a)\,\mathrm{Pf}\big[K(\vec{y};\vec{m})\big]$$

Generating function?

From the formula one can prove that $\mathbb{E}^{\text{flat}}\left[\tau^{\frac{1}{2}mh(t,0)}\right] \le c\tau^{-\frac{1}{4}m^2}$.
In fact, this bound should be close to being sharp, so
$\sum_{m\ge 0} \frac{1}{m!_\tau} \mathbb{E}^{\text{flat}}\left[\tau^{\frac{1}{2}mh(t,0)}\right]$ diverges.

Thus, if we want to sum the moments,
we are forced to premultiply by $\tau^{\frac{1}{4}m^2}$.

Additionally, premultiplying by $\tau^{\frac{1}{4}m^2}$ gets rid of $\prod_{1\le a<b\le k} \tau^{-\frac{1}{2}m_a m_b}$
and allows one to write the result as a Fredholm Pfaffian.

Flat ASEP moment formula

Recall the standard τ-exponential function

$$e_\tau(x) = \sum_{k=0}^{\infty} \frac{x^k}{k!_\tau},$$

where $k!_\tau = [k]_\tau [k-1]_\tau \cdots [2]_\tau [1]_\tau$ with $[j]_\tau = \frac{1-\tau^j}{1-\tau}$.

Flat ASEP moment formula

Instead we will use the

Symmetric τ-exponential fn.:

$$\exp_\tau(x) = \sum_{k=0}^{\infty} \tau^{\frac{k(k-1)}{4}} \frac{x^k}{k!_\tau},$$

which satisfies

$$\exp_\tau(x) = \exp_{\tau^{-1}}(x).$$

Flat ASEP moment formula

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$$\exp_\tau(x) = \exp_{\tau^{-1}}(x).$$

The result is that, for $\zeta \in \mathbb{C}$, $|\zeta| < \tau^{1/4}$,

$$\mathbb{E}^{\text{flat}}\left[\exp_\tau\left(-\zeta \tau^{\frac{1}{2}h(t,0)}\right)\right] = \mathrm{Pf}\left(J - K\right)_{L^2[0,\infty)}.$$

The Fredholm Pfaffian is defined as

$$\mathrm{Pf}\left(J-K\right)_{L^2[0,\infty)} = \sum_{n=0}^{\infty} \frac{(-1)^n}{n!} \int_{[0,\infty)^n} \mathrm{Pf}\left[K(x_i,x_j)\right]_{i,j=1}^{n} d\vec{x}$$

$$= \sqrt{\det(I+JK)_{L^2([0,\infty))\otimes L^2([0,\infty))}} \qquad J = \begin{pmatrix} 0 & I \\ -I & 0 \end{pmatrix}$$

$$K^{\text{ASEP}} = \begin{pmatrix} K_{1,1} & K_{1,2} \\ -K_{1,2} & K_{2,2} \end{pmatrix}$$

$$\begin{aligned} K_{1,1}(\lambda_1,\lambda_2) &= \sum_{m=1}^{\infty} \frac{1}{2\pi \mathrm{i}} \int_{C_{0,1}} dy\, \tau^{\frac{1}{2}m^2} \zeta^{2m} v(\lambda_1, y, m) v(\lambda_2, 1/y, m) u_1(y,m) \\ &\quad + \sum_{m_1,m_2=1}^{\infty} \sum_{\varsigma_1,\varsigma_2 \in \{-1,1\}} (-\varsigma_1\varsigma_2)^{m_1 \wedge m_2} \operatorname{sgn}(\varsigma_2 m_2 - \varsigma_1 m_1) u_2(\varsigma_1, m_1) u_2(\varsigma_2, m_2) \\ K_{1,2}(\lambda_1,\lambda_2) &= -\sum_{m=1}^{\infty} \sum_{\varsigma \in \{-1,1\}} \varsigma \tau^{\frac{1}{4}m^2} \zeta^m v(\lambda_1, \varsigma, m) u_2(\varsigma, m) \\ K_{2,2}(\lambda_1,\lambda_2) &= \tfrac{1}{2} \operatorname{sgn}(\lambda_2 - \lambda_1) \end{aligned}$$

with

$$v(\lambda, y, m) = \tfrac{1-\tau^{m/2}y}{1+\tau^{m/2}y} \exp\left(-\lambda \tfrac{1-\tau^{m/2}y}{1+\tau^{m/2}y} + t\left[\tfrac{1}{1+\tau^{-m/2}y} - \tfrac{1}{1+\tau^{-m/2}y}\right]\right) \times \tau^m (1-\tau)^m \frac{\left(-\tau^{-n/2}y;\tau\right)_\infty \left(\tau^{1+n}y^2;\tau\right)_\infty}{\left(-\tau^{n/2}y;\tau\right)_\infty \left(\tau y^2;\tau\right)_\infty},$$

$$u_1(y,m) = (-1)^m \tau^{-m} \tfrac{1+y^2}{y^2-1}, \qquad u_2(y,m) = \tau^{-\frac{1}{2}m} y \frac{1-\tau^m y^2}{1-\tau^m}.$$

By formal critical point analysis we can verify that, under the correct $t \to \infty$ scaling, $\mathrm{Pf}(J-K)_{L^2[0,\infty)}$ converges to

$$\mathrm{Pf}\left[J - \begin{pmatrix} \partial_{\lambda_2} K_{\mathrm{Ai}}(\lambda_1,\lambda_2) + \frac{1}{2}\mathrm{Ai}(\lambda_1)\,\mathrm{Ai}(\lambda_2) & \frac{1}{2}\mathrm{Ai}(\lambda_1) \\ -\frac{1}{2}\mathrm{Ai}(\lambda_2) & \frac{1}{2}\mathrm{sgn}(\lambda_1-\lambda_2) \end{pmatrix}\right] = F_{\mathrm{GOE}}(r),$$

as conjectured.

- The asymptotic analysis of $\mathrm{Pf}(J-K)_{L^2[0,\infty)}$ presents new technical difficulties.
- Additionally $\exp_\tau$ behaves very badly on $(-\infty,0)$ and in fact convergence of $\mathbb{E}^{\mathrm{flat}}\left[\exp_\tau\left(-\zeta\tau^{\frac{1}{2}h(t,0)}\right)\right]$ (suitably rescaled) does not hold.

Wishlist

- Rigorous $t \to \infty$ asymptotics.
- Rigorous weakly asymmetric limits.
- Multipoint distributions (even in the step case).
- Universality!