Degree and substructure in infinite graphs

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Abstract

We are interested in the relation between the average/minimum degree and the appearance of substructures in infinite graphs.

In finite graphs, the impact of the average degree on the appearance of certain substructures is well studied. The specific substructure that will play the lead role here is the complete graph K^k, for $k \in \mathbb{N}$, which may appear as a subgraph, as a minor, or as a topological minor. So a good example of a result in the direction we aim at is Turán's classical theorem, which states that an average degree of more than $\frac{k-2}{k-1}n$, where n is the number of vertices of the host graph G, ensures the existence of a finite subgraph of G that is isomorphic to K^k.

While the function from Turán's theorem depends on n, for forcing 'weaker' substructures the degree bound may depend only on k: An average degree of at least c_1k^2 ensures a topological minor isomorphic to K^k, and an average degree of at least $c_2k\sqrt{\log k}$ ensures a minor isomorphic to K^k. (The c_i are some constants from $\mathbb{R}_+$.)

Further, related substructures such as the complete k-partite graph K^k_s with partition classes of size s, or k-connected subgraphs, can be forced with stronger/weaker assumptions. In the following table, where we assume G to be a graph on n vertices, the reader finds an overview of some well-known results we would like to extend to infinite graphs. For the sake of brevity, in the first of these results, the quantifiers are missing: for every ε, k and s there is an n_0 so that for all $n \geq n_0$ the implication below is valid.

	Erdős-Stone	Turán	Bollobás & Thomason	Kostochka	Mader
$d(G)$	$> (\frac{k-2}{k-1} + \varepsilon)n$	$> \frac{k-2}{k-1}n$	$\geq c_1k^2$	$\geq c_2k\sqrt{\log k}$	$\geq 4k$
$\Rightarrow$	$K^k_s \subseteq G$	$K^k \subseteq G$	$K^k \preceq_{top} G$	$K^k \preceq G$	$H \subseteq G$, H $(k+1)$-connected.

So how do these results extend to infinite graphs? To answer this question, we must ask first of all how the average degree translates to an infinite graph, as we now deal with an infinite number of vertices. Of course, the average degree

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is closely related to the density $|E(G)|/\binom{|V(G)|}{2}$, and this notion is reflected in the *upper density* of an infinite graph.

The upper density $ud(G)$ of a graph G is defined as the supremum of the subgraph densities, taken over all sequences of finite subgraphs of G whose order tends to infinity. That is, $ud(G) := \sup_{(H_i)_{i\in\mathbb{N}}} \limsup_{i\to\infty}(|E(H_i)|/\binom{V(H_i)}{2})$, where the sequences $(H_i)_{i\in\mathbb{N}}$ range over all sequences of finite subgraphs $H_i \subseteq G$ with $\lim_{i\to\infty}|V(H_i)| = \infty$ (see for instance Bollobás [1]).

Now, it is not difficult to calculate that if (for $k > 1$) the upper density of a graph G is greater than $\frac{k-2}{k-1}$, say $ud(G) \geq (1+\delta)\frac{k-2}{k-1}$, then G has a finite subgraph H of average degree at least $(1+\frac{\delta}{2})\frac{k-2}{k-1}|V(H)|$, and thus, by Turán's theorem, contains a K^k-subgraph. Actually, as the order of the subgraph H may be assumed to exceed any given integer, we may apply to H the Erdős-Stone theorem for any s, and obtain a K^k_s-subgraph. So in this sense, both the Turán and the Erdős-Stone theorem do extend to infinite graphs.

From the existence of arbitrarily large complete k-partite subgraphs once the threshold upper density $\frac{k-2}{k-1}$ is surpassed, it follows that the upper density of any infinite graph takes one of the following (countably many) values: $1, 0, \frac{1}{2}, \frac{2}{3}, \frac{3}{4}, \ldots$, that is, one of the Turán densities. So it seems that the graphs for which it would be interesting to extend the latter three results discussed above, all have upper density 0. In other words, the upper density is not fine enough a measure for a generalisation of e.g. Kostochka's theorem to infinite graphs.

One possible way out of this dilemma is replacing the average degree with something that quite obviously does exist in infinite graphs, the minimum degree. For rayless graphs, this is an excellent option, as we have the following result, which is not difficult to prove. Write $\delta^V(G)$ for the minimum degree taken over all vertices of the graph G.

Proposition 1. [4] *Let $k \in \mathbb{N}$ and let G be a rayless graph with $\delta^V(G) \geq k$. Then G has a finite subgraph of average degree at least k.*

This means that the latter three results from the table above extend literally to rayless graphs, if we replace the average degree with the minimum degree.

In general, however, we are not that lucky. Just consider an infinite tree, whose vertices may attain any minimal degree condition, while the tree does not contain any interesting substructure. The example suggests that we need some additional condition that prevents 'the density from escaping to infinity', in other words, that makes the vertices send their edges 'back' instead of 'further out'. Following recent developments (see [3]), the most natural way to impose such an additional condition is to impose it on the ends[1] of the graph.

In [2, 5], see also [3], the *vertex-degree*[2] $d_v(\omega)$ of an end ω is defined as the supremum of the cardinalities of the sets of vertex-disjoint rays from ω. This intuitive notion allows us to extend Mader's theorem from above to infinite

[1] The *ends* of a graph are the equivalence classes of the rays (the one-way infinite paths) of the graph under the following equivalence relation. Two rays are equivalent if no finite set of vertices separates them. For more on the end space of an infinite graph, see [3].

[2] There, also the *edge-degree* of ω is defined quite analogously, as the supremum of the cardinalities of the sets of edge-disjoint rays from ω. The edge-degree allows for an extension of the edge-version of Mader's theorem. In this edge-version, only linear bounds on the minimum (edge)-degree are needed. For details, see [5].

graphs. For this, let us write $\delta^{V,\Omega_v}(G)$ for the minimum of the degrees or vertex-degrees, taken over all vertices and ends of the graph G.

Theorem 2. **[5]** *Let G be a graph. If $\delta^{V,\Omega_v}(G) \geq 2k(k+3)$ then G has a $(k+1)$-connected subgraph.*

We remark that the $(k+1)$-connected subgraph can neither be guaranteed to be finite nor to be infinite. The bound $2k(k+3)$ may possibly be lowered, but not to less than $\frac{k}{5}\log\frac{k}{5}$. See [5].

The vertex-degree, however, does not serve for forcing large complete (topological) minors. One can see this by considering the following example. Take, for $r \in \mathbb{N}$, $r > 2$, the infinite r-regular tree, and add a spanning cycle in each level. The resulting graph G_r has one end of infinite vertex-degree, while all vertices have degree at least r. Now, although r may be arbitrarily large, G_r is planar, and thus has no complete minor of order greater than 4.

So, a different road has to be taken for forcing minors and topological minors in graphs with rays. In [4], the *relative degree* of an end was introduced for locally finite graphs. The idea is to calculate the ratios of the cardinality of the edge-boundary[3] $\partial_e H_i$ versus the cardinality of the vertex-boundary[4] $\partial_v H_i$ of certain subgraphs H_i of G, and then define the relative degree to be the limit of these ratios as the H_i in some sense converge to ω. This intuitive idea can be formalised as follows.

We call a subgraph H of a graph G an *ω-region* if $\partial_v H$ is finite and H contains a ray of the end $\omega \in \Omega(G)$. We write $\Omega^G(H)$ for the sets of all ends of G that have a ray in H.

Now, for a locally finite graph G, write $(H_i)_{i\in\mathbb{N}} \to \omega$ if $(H_i)_{i\in\mathbb{N}}$ is an infinite sequence of distinct ω-regions of G such that $H_{i+1} \subseteq H_i - \partial_v H_i$ and $\partial_v H_{i+1}$ is an inclusion-minimal $\partial_v H_i$–$\Omega^G(H_{i+1})$ separator, for each $i \in \mathbb{N}$. Note that by the local finiteness of G such sequences do exist. Define

$$d_{e/v}(\omega) := \inf_{(H_i)_{i\in\mathbb{N}}\to\omega} \liminf_{i\to\infty} \frac{|\partial_e H_i|}{|\partial_v H_i|}.$$

This definition leads to the desired results for locally finite graphs. Let $\delta^{V,\Omega_{e/v}}(G)$ denote the minimum (relative) degree, taken over all vertices and ends of the graph G. The constants $c_1, c_2 \in \mathbb{R}_+$ are as in the corresponding theorems for finite graphs.

Theorem 3. **[4]** *Let $k \in \mathbb{N}$ and let G be a locally finite graph.*

(a) If $\delta^{V,\Omega_{e/v}}(G) \geq c_1 k^2$, then K^k is a topological minor of G.

(b) If $\delta^{V,\Omega_{e/v}}(G) \geq c_2 k\sqrt{\log k}$, then K^k is a minor of G.

For arbitrary infinite graphs it is necessary to adapt the definition of the relative degree. This is so as now there may be vertices dominating[5] ends. In that case, the sequences $(H_i)_{i\in\mathbb{N}}$ cannot satisfy the condition that $H_{i+1} \subseteq H_i - \partial_v H_i$. We thus ask:

[3]The *edge-boundary* of a subgraph H of a graph G is the set $\partial_e H := E(H, G-H)$.

[4]The *vertex-boundary* of a subgraph H of a graph G is the set $\partial_v H := N_G(G-H)$.

[5]A vertex is said to *dominate* an end ω if for some ray $R \in \omega$ there are infinitely many v–$V(R)$ paths, disjoint except in v.

Question 4. *Does Theorem 3 extend to arbitrary infinite graphs? How does the (relative) end degree have to be defined in this case?*

A partial answer to Question 4 will be provided in [6]. A second not less interesting question is:

Question 5. *Are there extensions of the results mentioned in the beginning, if we let k be an infinite cardinal?*

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