Inducing an order on cellular automata by a grouping operation [1]

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Abstract

Let X be a one-dimensional cellular automaton. A "power of X" is another cellular automaton obtained by grouping several states of X into blocks and by considering as local transitions the "natural" interactions between neighbor blocks. Based on this operation a preorder $\leq$ on the set of one-dimensional cellular automata is introduced. We denote by (CA*,$\leq$) the canonical order induced by $\leq$. We prove that (CA*,$\leq$) admits a global minimum and that very natural equivalence classes are located at the bottom of (CA*,$\leq$). These classes remind us the first two well-known Wolfram ones because they capture global (or dynamical) properties as nilpotency or periodicity. Non-trivial properties as the undecidability of $\leq$ and the existence of bounded infinite chains are also proved. Finally, it is shown that (CA*,$\leq$) admits no maximum. This result allows us to conclude that, in a "grouping sense", there is no universal CA.

Key words: cellular automata; grouping; dynamical classification; intrinsic universality; order.

1 Introduction

One-dimensional cellular automata with radius 1, or simply CA, are infinite arrays of finite-state machines called cells and indexed by $\mathbb{Z}$. These identical cells evolve synchronously at discrete time steps following a local rule by which the state of a cell is determined as a function of its own state together with the states of its two neighbors. These devices, despite their simplicity, may exhibit very complex behavior.

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In order to "understand" these CA one should find some criteria capable of structuring them into natural classes or hierarchies. In this direction, the classification of S. Wolfram [11], though heuristical and coarse, corresponds to the best-known attempt. Wolfram, by "observing" the long-time behavior of "arbitrary" periodic configurations, distinguishes four CA classes. Some efforts have been made in order to formalize this classification [5] or, typically by dynamical systems arguments, to introduce new classification schemes [3,2]. Unfortunately, this last approach yields to some paradoxes: the shift CA, for instance, appears to be chaotic.

CA may also be seen as computational devices. In fact, it is easy to exhibit a CA that simulates any Turing machine [6]. In other words, the CA model is Turing-universal. The question whether the CA model is intrinsic-universal or, more precisely, whether there exists a CA capable of simulating any other, remained open for some years. Notice that the CA can not be simulated by a Turing machine because the latter has a unique head which obviously will never visit the whole tape. J. Albert and K. Čulik II exhibited in [1] an intrinsic-universal CA. The "intrinsic-reducibility" notion induces a preorder on the set of CA. There are no results concerning (the structure of) this preorder and, in addition, its study seems to be very difficult: it is based by definition on the evolution of all the possible configurations (which are uncountable) and it does not take explicitly into account the CA transition tables. On the other hand, the "intrinsic simulation" notion is so broad that a pair of CA with extremely different dynamics could appear to be "equivalent".

Another approach is to consider CA as algebraic objects. In this context, with the purpose of endowing the set of CA with an order relation, it would be sufficient to say that A is a subautomaton of B if the transition table of A is contained (after a suitable relabeling of the states) in the transition table of B. This notion is extremely restrictive. In fact, if A is a subautomaton of B then the space-time diagrams of A are "cell by cell equivalent" to the corresponding space-time diagrams of B (space-time diagrams are representations in $\mathbb{Z}^2$ of a CA evolution from a particular initial configuration). In other words, A and B may not be associated by the subautomaton relation even with their respective space-time diagrams being identical after suitable "changes of scale".

It seems therefore very natural to try to replace the subautomaton relation by a new one which could take into account potential changes of scale. This can be done by defining the powers of a CA. More precisely, let us denote by X^i the CA that generates the i-scaled space-time diagram of X and which is simply obtained by grouping i cells (or states) into blocks and by considering as transitions the interactions of neighbor blocks. Let us also write $A \leq B$ when some power of A is a subautomaton of some power of B or, equivalently, when the space-time diagrams of A are "block by block equivalent" to the corresponding space-time diagrams of B. In Section 2 all these definitions are formally given.

In Section 3 we prove that $\leq$ is a preorder on CA. We denote by CA* the set of the canonical equivalence classes induced by $\leq$, and we show some basic properties concerning the order (CA*,$\leq$). In particular, we prove that (CA*,$\leq$) admits a global minimum.

In Section 4 we exhibit some equivalence classes located at the bottom of (CA*,$\leq$). These classes, besides being located immediately above the global minimum, appear to be very natural. In fact, they remind us the first two well-known Wolfram ones because they capture dynamical properties as nilpotency or periodicity.

In Section 5 we prove a non-trivial property concerning (CA*,$\leq$): the existence of two incomparable infinite chains having a common upper bound. This upper bound corresponds to the equivalence class represented by a "synchronization CA". Notice that it could be said that the order (CA*,$\leq$) "takes into account the algorithmical non-triviality of this synchronization CA" because it admits (at least) a pair of infinite chains separating it from the minimum.

Finally, in Section 6 we prove that (CA*,$\leq$) has no maximum. Moreover, we prove that even maximal elements do not exist in (CA*,$\leq$). In other words, in a "grouping sense", there is no universal CA. This result gives us a lower-bound in the more general framework of "intrinsic-universality on CA" developed by Albert and Čulik II [1].

2 Definitions

Formally, a CA is determined by a couple (Q, δ) where Q is a finite set of states and $\delta : Q^3 \to Q$ is a transition function. A configuration of a CA (Q, δ) is a bi-infinite sequence $\mathcal{C} \in Q$, and its global transition function $G_\delta : Q \quad \to Q$ is such that $(G_\delta(\mathcal{C}))_i = \delta(\mathcal{C}_{i-1}, \mathcal{C}_i, \mathcal{C}_{i+1})$. For $t \in \mathbb{N}^* = \mathbb{N} \setminus \{0\}$ it is defined recursively $G_\delta^t(\mathcal{C}) = G_\delta(G_\delta^{(t-1)}(\mathcal{C}))$ with $G_\delta^0(\mathcal{C}) = \mathcal{C}$.

We say that (Q_1, δ_1) is a subautomaton of (Q_2, δ_2), and we write $(Q_1, \delta_1) \subseteq (Q_2, \delta_2)$, if there exists an injection $\varphi : Q_1 \to Q_2$ such that for all $x, y, z \in Q_1$:

$$\varphi(\delta_1(x, y, z)) = \delta_2(\varphi(x), \varphi(y), \varphi(z)).$$

When the function φ is a bijection we say that (Q_1, δ_1) and (Q_2, δ_2) are isomorphic and we write $(Q_1, \delta_1) \cong (Q_2, \delta_2)$.

For any CA (Q, δ) the evolution of a finite block of states looks like a light-cone

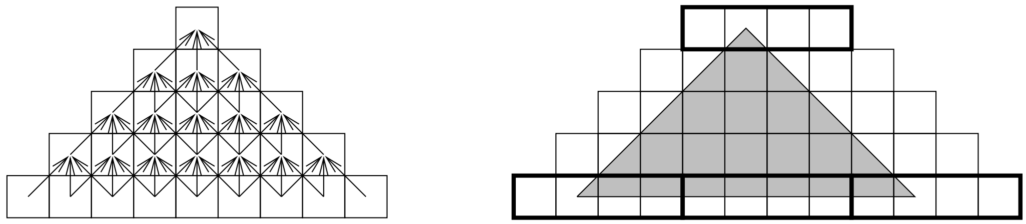

Fig. 1. (i) Dependencies diagram representing a block of states evolution as a light-cone. (ii) Interaction of three blocks.

(see Figure 1-i). This basic fact inspires the notion of the n-block evolution function $\delta^n : Q^{2n+1} \to Q$, which is recursively defined for all $n \in \mathbb{N}^*$ as follows:

$$\begin{aligned}\delta^1(w_{-1}, w_0, w_1) &= \delta(w_{-1}, w_0, w_1),\\ \delta^n(w_{-n} \cdots w_0 \cdots w_n) &= \delta^{n-1}(\delta(w_{-n}, w_{-n+1}, w_{-n+2}) \cdots \delta(w_{n-2}, w_{n-1}, w_n)).\end{aligned}$$

By grouping several states into blocks and by letting interact triplets of blocks as schematically appears in Figure 1-ii, we generate CA with (exponentially) more states.

Formally, the n-power of a CA (Q, δ) is the CA $(Q, \delta)^n = (Q^n, \delta^n_{\mathcal{G}})$, where $\vec{q} \in Q^n$ is denoted by $(q_1 \cdots q_n)$ and for all $\vec{x}, \vec{y}, \vec{z} \in Q^n$:

$$(\delta^n_{\mathcal{G}}(\vec{x}, \vec{y}, \vec{z}))_i = \delta^n(x_i \cdots x_n y_1 \cdots y_i \cdots y_n z_1 \cdots z_i).$$

Let us define the relation $\leq$ in such a way that it associates two CA when some power of the first is a subautomaton of some power of the second. More precisely, for any pair of CA (Q_1, δ_1) and (Q_2, δ_2):

$$(Q_1, \delta_1) \leq (Q_2, \delta_2) \iff \exists n, m \in \mathbb{N}^* : (Q_1, \delta_1)^n \subseteq (Q_2, \delta_2)^m.$$

3 An order on CA

Here we show that the relation $\leq$ is a preorder on CA. We denote by CA* the set of the canonical equivalence classes induced by $\leq$, and we prove some basic properties concerning the order (CA*,$\leq$). In fact, we first show that at least all the powers of a CA belong to the same equivalence class. Then we prove that (CA*,$\leq$) admits a global minimum consisting of all the isomorphic CA having a single state. Finally it is shown that every finite family of CA* admits a maximum obtained by a simple "superposition operation".

The following lemmas are formulated just for proving Proposition 1 (which states that $\leq$ is a preorder on CA).

Lemma 1 *Let (Q,δ) be a CA. For all $n>1$:*

$$\begin{aligned}\delta^n(w_{-n}\cdots w_n) = \delta(&\delta^{n-1}(w_{-n}\cdots w_{n-2}),\\ &\delta^{n-1}(w_{-n+1}\cdots w_{n-1}),\\ &\delta^{n-1}(w_{-n+2}\cdots w_n)).\end{aligned}$$

Proof. By induction on n. For $n=2$ it is direct by definition. Assuming it true for n and denoting $A=\delta^{n+1}(w_{-n-1}\cdots w_{n+1})$, it follows:

$$\begin{aligned}A &= \delta^n(\delta(w_{-n-1},w_{-n},w_{-n+1})\cdots\delta(w_{n-1},w_n,w_{n+1}))\\ &= \delta(\delta^{n-1}(\delta(w_{-n-1},w_{-n},w_{-n+1})\cdots\delta(w_{n-3},w_{n-2},w_{n-1})),\\ &\quad\ \delta^{n-1}(\delta(w_{-n},w_{-n+1},w_{-n+2})\cdots\delta(w_{n-2},w_{n-1},w_n)),\\ &\quad\ \delta^{n-1}(\delta(w_{-n+1},w_{-n+2},w_{-n+3})\cdots\delta(w_{n-1},w_n,w_{n+1})))\\ &= \delta(\delta^n(w_{-n-1}\cdots w_{n-1}),\delta^n(w_{-n}\cdots w_n),\delta^n(w_{-n+1}\cdots w_{n+1})).\quad\square\end{aligned}$$

Lemma 2 *Let (Q,δ) be a CA. For all $i,n\in\mathbb{N}^*$ such that $i<n$:*

$$\delta^n(w_{-n}\cdots w_n)=\delta^{n-i}(\delta^i(w_{-n}\cdots w_{-n+2i})\cdots\delta^i(w_{n-2i}\cdots w_n)).$$

Proof. By induction on i. For $i=1$ it is direct by definition. Assuming it true for i, considering Lemma 1 and denoting $A=\delta^n(w_{-n}\cdots w_n)$, it follows:

$$\begin{aligned}A &= \delta^{n-i}(\delta^i(w_{-n}\cdots w_{-n+2i})\cdots\delta^i(w_{n-2i}\cdots w_n))\\ &= \delta^{n-i-1}(\delta^{i+1}(w_{-n}\cdots w_{-n+2i+2})\cdots\delta^{i+1}(w_{n-2i-2}\cdots w_n)).\quad\square\end{aligned}$$

Lemma 3 *Let $(Q_1,\delta_1)\subseteq(Q_2,\delta_2)$. For all $n\in\mathbb{N}^*$:*

$$(Q_1,\delta_1)^n\subseteq(Q_2,\delta_2)^n.$$

Proof. Let $\varphi:Q_1\to Q_2$ be a suitable injection. First we have to prove by induction on n that $\varphi(\delta_1^n(w_{-n}\cdots w_n))=\delta_2^n(\varphi(w_{-n})\cdots\varphi(w_n))$. It is direct for $n=1$. Assuming it true for n, and denoting $A=\varphi(\delta_1^{n+1}(w_{-n-1}\cdots w_{n+1}))$, it follows:

$$\begin{aligned}A &= \varphi(\delta_1^n(\delta_1(w_{-n-1},w_{-n},w_{-n+1})\cdots\delta_1(w_{n-1},w_n,w_{n+1})))\\ &= \delta_2^n(\varphi(\delta_1(w_{-n-1},w_{-n},w_{-n+1}))\cdots\varphi(\delta_1(w_{n-1},w_n,w_{n+1})))\\ &= \delta_2^n(\delta_2(\varphi(w_{-n-1}),\varphi(w_{-n}),\varphi(w_{-n+1}))\cdots\delta_2(\varphi(w_{n-1}),\varphi(w_n),\varphi(w_{n+1})))\\ &= \delta_2^{n+1}(\varphi(w_{n-1})\cdots\varphi(w_{n+1})).\end{aligned}$$

Let us consider now the injection $\vec{\varphi}:(Q_1)^n\to(Q_2)^n$ such that $\vec{\varphi}(\vec{x})=(\varphi(x_1)\cdots\varphi(x_n))$. It follows that for all $\vec{x},\vec{y},\vec{z}\in(Q_1)^n$:

$$\begin{aligned}(\vec{\varphi}((\delta_1)^n_{\mathcal{G}}(\vec{x},\vec{y},\vec{z}))_i &= \varphi(\delta_1^n(x_i\cdots y_i\cdots z_i))\\ &= \delta_2^n(\varphi(x_i)\cdots\varphi(y_i)\cdots\varphi(z_i))\\ &= ((\delta_2)^n_{\mathcal{G}}(\vec{\varphi}(\vec{x}),\vec{\varphi}(\vec{y}),\vec{\varphi}(\vec{z}))_i. \qquad \square\end{aligned}$$

Lemma 4 *Let (Q,δ) be a CA. For all $n,m\in\mathbb{N}^*$:*

$$((Q,\delta)^n)^m \cong (Q,\delta)^{nm}.$$

Proof. Writing $\vec{a}\in(Q^n)^m$ by $(\vec{a}_1\cdots\vec{a}_m)$ with $\vec{a}_i=(a_{i1}\cdots a_{in})\in Q^n$ and $\vec{b}\in Q^{nm}$ by $(b_{11}\cdots b_{1n}\cdots\cdots b_{m1}\cdots b_{mn})$ with $b_{ij}\in Q$, and defining the bijection $\varphi:(Q^n)^m\to Q^{nm}$ such that $(\varphi(\vec{a}))_{ij}=(\vec{a}_i)_j$, in order to prove the lemma it suffices to show that the next identity holds for all $i\le m, j\le n$, and for all $\vec{x},\vec{y},\vec{z}\in(Q^n)^m$:

$$[(\delta^n_{\mathcal{G}})^m(\vec{x}_i\cdots\vec{x}_m\vec{y}_1\cdots\vec{y}_m\vec{z}_1\cdots\vec{z}_i)]_j$$
$$=$$
$$\delta^{nm}((\vec{x}_i)_j\cdots(\vec{x}_m)_n(\vec{y}_1)_1\cdots(\vec{y}_m)_n(\vec{z}_1)_1\cdots(\vec{z}_i)_j),$$

because

$$\begin{aligned}[\varphi((\delta^n_{\mathcal{G}})^m_{\mathcal{G}}(\vec{x},\vec{y},\vec{z}))]_{ij} &= [((\delta^n_{\mathcal{G}})^m_{\mathcal{G}}(\vec{x},\vec{y},\vec{z}))_i]_j\\ &= [(\delta^n_{\mathcal{G}})^m(\vec{x}_i\cdots\vec{x}_m\vec{y}_1\cdots\vec{y}_m\vec{z}_1\cdots\vec{z}_i)]_j,\\ \text{and } [\delta^{nm}_{\mathcal{G}}(\varphi(\vec{x}),\varphi(\vec{y}),\varphi(\vec{z}))]_{ij} &= \delta^{nm}((\varphi(\vec{x}))_{ij}\cdots(\varphi(\vec{z}))_{ij})\\ &= \delta^{nm}((\vec{x}_i)_j\cdots(\vec{z}_i)_j).\end{aligned}$$

We finally prove the identity by induction on m. For $m=1$ it holds directly. Let us assume it true for m. It follows:

$$\begin{aligned}[(\delta^n_{\mathcal{G}})^{m+1}(\vec{x}_i\cdots\vec{z}_i)]_j &= [(\delta^n_{\mathcal{G}})^m(\delta^n_{\mathcal{G}}(\vec{x}_i,\vec{x}_{i+1},\vec{x}_{i+2})\cdots\delta^n_{\mathcal{G}}(\vec{z}_{i-2},\vec{z}_{i-1},\vec{z}_i))]_j\\ &= \delta^{nm}((\delta^n_{\mathcal{G}}(\vec{x}_i,\vec{x}_{i+1},\vec{x}_{i+2}))_j\cdots(\delta^n_{\mathcal{G}}(\vec{z}_{i-2},\vec{z}_{i-1},\vec{z}_i))_j)\\ &= \delta^{nm}(\delta^n((\vec{x}_i)_j\cdots(\vec{x}_{i+2})_j)\cdots\delta^n((\vec{z}_{i-2})_j\cdots(\vec{z}_i)_j))\\ &= \delta^{n(m+1)}((\vec{x}_i)_j\cdots(\vec{z}_i)_j). \quad \square\end{aligned}$$

Proposition 1 *The relation $\le$ is a preorder on CA.*

Proof. The reflexivity holds directly. For the transitivity, let us consider $(Q_1,\delta_1)\le(Q_2,\delta_2)$ and $(Q_2,\delta_2)\le(Q_3,\delta_3)$. By definition, there exist n_1, m_1, n_2, $m_2\in\mathbb{N}^*$ such that $(Q_1,\delta_1)^{n_1}\subseteq(Q_2,\delta_2)^{m_1}$ and $(Q_2,\delta_2)^{n_2}\subseteq(Q_3,\delta_3)^{m_2}$. By applying Lemma 3 and Lemma 4, together with the transitivity of $\subseteq$, we conclude that $(Q_1,\delta_1)^{n_1n_2}\subseteq(Q_3,\delta_3)^{m_1m_2}$. $\square$

Remark 1 *As any other preorder, the relation $\leq$ induces:*

- *An equivalence relation $\sim$ on CA, with $(Q_1, \delta_1) \sim (Q_2, \delta_2)$ if and only if $(Q_1, \delta_1) \leq (Q_2, \delta_2)$ and $(Q_2, \delta_2) \leq (Q_1, \delta_1)$.*
- *A strict preorder $<$ on CA, with $(Q_1, \delta_1) < (Q_2, \delta_2)$ if and only if $(Q_1, \delta_1) \leq (Q_2, \delta_2)$ and $(Q_1, \delta_1) \not\sim (Q_2, \delta_2)$.*
- *The canonical order on $(CA/\sim)$ compatible with $\leq$ and denoted by $(CA^*, \leq)$.*

Proposition 2 *For any CA (Q, δ), all its powers are equivalent. In other words, for all $i, j \in \mathbb{N}^*$: $(Q, \delta)^i \sim (Q, \delta)^j$.*

Proof. By Lemma 4, $((Q, \delta)^j)^i \cong (Q, \delta)^{ij} \cong ((Q, \delta)^i)^j$. $\Box$

Definition 1 *Let us denote by $SING$ the family of all the CA having a single state. More precisely,*

$$SING = \{(Q, \delta) : |Q| = 1\}.$$

Proposition 3 *$SING$ corresponds to the global minimum of $(CA^*, \leq)$.*

Proof. First notice that all the CA of $SING$ are isomorphic. Let $(\{s\}, \delta_s) \in SING$ and let (Q, δ) be an arbitrary CA. By the finiteness of Q there exist $\tilde{q} \in Q$ and $P \in \mathbb{N}^*$ with $1 \leq P \leq |Q|$ such that $\delta^P(\tilde{q}, \cdots, \tilde{q}) = \tilde{q}$, and therefore $(\{s\}, \delta_s) \subseteq (Q, \delta)^P$. Finally notice that if $|Q| > 1$, then $(\{s\}, \delta_s) < (Q, \delta)$ because any power of a singleton CA is also a singleton CA. $\Box$

Definition 2 *Let $(Q_1, \delta_1), (Q_2, \delta_2)$ be two CA. We define the superposition $(Q, \delta) = (Q_1, \delta_1) \otimes (Q_2, \delta_2)$ as follows:*

- *$Q = (Q_1 \cup \{B\}) \times (Q_2 \cup \{B\})$ with B not being a state of any CA.*
- *For all $\vec{x} = (x_1, x_2), \vec{y} = (y_1, y_2), \vec{z} = (z_1, z_2) \in Q$:*

$$\delta(\vec{x}, \vec{y}, \vec{z}) = \begin{cases} (\delta_1(x_1, y_1, z_1), B) & \textit{if } (\vec{x}, \vec{y}, \vec{z}) \in (Q_1 \times \{B\})^3, \\ (B, \delta_2(x_2, y_2, z_2)) & \textit{if } (\vec{x}, \vec{y}, \vec{z}) \in (\{B\} \times Q_2)^3, \\ (B, B) & \textit{otherwise.} \end{cases}$$

Proposition 4 *The order $(CA^*, \leq)$ admits local maxima. In other words, for every finite family $\{X_i\}_{i=1}^n \subseteq CA^*$ there exists $Y \in CA^*$ such that $X_i \leq Y$ for all $i \in \{1, \cdots, n\}$.*

Proof. It suffices to notice that for all (Q_1, δ_1) and (Q_2, δ_2), the superposition $(Q, \delta) = (Q_1, \delta_1) \otimes (Q_2, \delta_2)$ satisfies $(Q_1, \delta_1) \leq (Q, \delta)$ and $(Q_2, \delta_2) \leq (Q, \delta)$. This fact can be easily proved by considering the injections $\varphi_1 : Q_1 \to Q$ with $\varphi(x_1) = (x_1, B)$ and $\varphi_2 : Q_2 \to Q$ with $\varphi(x_2) = (B, x_2)$. $\Box$

4 The bottom of (CA*,≤)

One expects the "simplest" CA to be located at the bottom of (CA*,≤). In this section we give some "evidence" supporting this intuitive expectation. In fact, we first show that the classes represented by the CA having trivial transition functions (constant, identity, shift) are all located at the bottom of (CA*,≤). In other words, there is nothing between them and the global minimum $SING$. Notice that, in addition, these classes are very natural: they remind us the first two well-known Wolfram ones [11] because they capture dynamical properties as nilpotency or periodicity.

Formally, a CA class is said to belong to the bottom of (CA*,≤) if there is no other CA class located strictly between the minimum and itself. In other words, a class represented by X belongs to the bottom of (CA*,≤) if for any other non-singleton CA Y: $Y \leq X \Longrightarrow X \leq Y$.

4.1 Nilpotency

The limit set is a fundamental concept of dynamical system theory. It corresponds "to the set of all the configurations that can occur after arbitrarily many computation steps". For a CA (Q, δ) we denote its limit set by $\Omega(Q, \delta)$. More precisely, if we define $\Omega^0 = Q$ and $\Omega^i = G_\delta(\Omega^{i-1})$ for $i \geq 1$, then $\Omega(Q, \delta) = \bigcap_{i=1}^{\infty} \Omega^i$. We say that a CA belongs to the class NIL, and we call it nilpotent, if its limit set is a singleton. In other words,

$$NIL = \{(Q, \delta) : (|Q| > 1) \wedge (|\Omega(Q, \delta)| = 1)\}.$$

Obviously, when the limit set is a singleton it corresponds to an homogeneous configuration. In [4] it is proved that when nilpotency holds then this configuration is reached from any other one in a finite and fixed number of steps. More precisely, if we denote $\overline{s}_0 = (\cdots s_0 s_0 s_0 \cdots)$:

$$NIL = \{(Q, \delta) : (|Q| > 1) \wedge (\exists s_0 \in Q, n \in \mathbb{N}^*)(\forall \mathcal{C} \in Q \quad)(G_\delta^n(\mathcal{C}) = \overline{s}_0)\}.$$

We introduce now the simplest nilpotent CA: those reaching the homogeneous configuration in one step. More precisely:

Definition 3 *Let the family of CA $\{(S_n, 0_n)\}_{n>1}$ be such that, for all $n > 1$:*

- $S_n = \{0, \cdots, n-1\}$.
- $\forall x, y, z \in S_n : 0_n(x, y, z) = 0$.

Lemma 5 *For all $n > 1$: $(S_2, 0_2) \sim (S_n, 0_n)$.*

Proof. First notice that if $p \leq q$ then $0_p = 0_q \big|_{S_p}$, and therefore $(S_p, 0_p) \subseteq (S_q, 0_q)$. Let $n > 1$ and let $\tilde{n} \in \mathbb{N}^*$ be such that $n \leq 2^{\tilde{n}}$. It follows that $(S_n, 0_n) \subseteq (S_2, 0_2)^{\tilde{n}}$ because $(S_2, 0_2)^{\tilde{n}} \cong (S_{2^{\tilde{n}}}, 0_{2^{\tilde{n}}})$. □

Lemma 6 *If $(Q, \delta) \leq (S_2, 0_2)$ then $(Q, \delta) \in NIL$ or $|Q| = 1$.*

Proof. If $(Q, \delta) \leq (S_2, 0_2)$ then $\exists i, j \in \mathbb{N}^* : (Q, \delta)^i \subseteq (S_2, 0_2)^j$. It follows:

$$
\begin{aligned}
& [\exists i, j \in \mathbb{N}^*][(Q, \delta)^i \subseteq (S_{2^j}, 0_{2^j})]. \\
\Longrightarrow\ & [\exists i \in \mathbb{N}^*, \vec{s} \in Q^i][\forall \vec{c}_1, \vec{c}_2, \vec{c}_3 \in Q^i][\delta^i_{\mathcal{G}}(\vec{c}_1, \vec{c}_2, \vec{c}_3) = \vec{s}]. \\
\Longrightarrow\ & [\exists i \in \mathbb{N}^*, s_0 \in Q][\forall \vec{c} \in Q^{2i+1}][(\delta^i(\vec{c}) = s_0]. \\
\Longrightarrow\ & (Q, \delta) \in NIL \vee |Q| = 1.
\end{aligned}
$$

□

Proposition 5 *The equivalence class represented by $(S_2, 0_2)$ corresponds to the family of nilpotent CA. In other words, for any CA (Q, δ):*

$$(Q, \delta) \sim (S_2, 0_2) \Longleftrightarrow (Q, \delta) \in NIL.$$

Proof. If $(Q, \delta) \in NIL$ then, by definition, there exist $n \in \mathbb{N}^*$ and $s_0 \in Q$ such that for all $\vec{c} \in Q^{2n+1} : \delta^n(\vec{c}) = s_0$. It follows that $(Q, \delta)^n \cong (S_{|Q|^n}, 0_{|Q|^n})$ with $|Q|^n > 1$ and therefore, by Lemma 5, $(Q, \delta) \sim (S_2, 0_2)$. The other implication corresponds to Lemma 6 and to the fact that if $(S_2, 0_2) \leq (Q, \delta)$ then $|Q| > 1$. □

Proposition 6 *The equivalence class NIL belongs to the bottom of $(CA^*, \leq)$. In other words, for any non-singleton CA (Q, δ):*

$$(Q, \delta) \leq (S_2, 0_2) \Longrightarrow (S_2, 0_2) \leq (Q, \delta).$$

Proof. Let $(Q, \delta) \leq (S_2, 0_2)$ with $|Q| > 1$. By Lemma 6, $(Q, \delta) \in NIL$, and therefore $(Q, \delta) \sim (S_2, 0_2)$. □

Corollary 1 *Given a CA (Q, δ), it is undecidable whether $(Q, \delta) \leq (S_2, 0_2)$.*

Proof. By the fact that the nilpotency problem is undecidable [7]. □

4.2 Periodicity and shift-like behavior

Now some other simple global properties concerning cyclic behavior are considered. First we say that a CA belongs to the class PER, and we call it periodic, if every configuration belongs to a cycle. More precisely,

$$PER = \{(Q,\delta) : (|Q| > 1) \wedge (\forall \mathcal{C} \in Q \ \ , \exists n \in \mathbb{N}^* : G_\delta^n(\mathcal{C}) = \mathcal{C})\}.$$

On the other hand we introduce the R_{SHIFT} and L_{SHIFT} classes. In this case, for every configuration there exists an $n \in \mathbb{N}^*$ for which the configuration reappears n cells shifted after n time steps. In other words,

$$R_{SHIFT} = \{(Q,\delta) : (|Q| > 1) \wedge (\forall \mathcal{C} \in Q \ \ , \exists n \in \mathbb{N}^* : ((G_\delta)^n(\mathcal{C}))_i = \mathcal{C}_{i-n})\}.$$
$$L_{SHIFT} = \{(Q,\delta) : (|Q| > 1) \wedge (\forall \mathcal{C} \in Q \ \ , \exists n \in \mathbb{N}^* : ((G_\delta)^n(\mathcal{C}))_i = \mathcal{C}_{i+n})\}.$$

As in the nilpotency case, for these classes the length of the cycles does not depend on the considered configurations. This result is stated in the next lemma:

Lemma 7 *The following holds:*

$$PER = \{(Q,\delta) : (|Q| > 1) \wedge (\exists n \in \mathbb{N}^*, \forall \mathcal{C} \in Q \ \ : G_\delta^n(\mathcal{C}) = \mathcal{C})\}.$$
$$R_{SHIFT} = \{(Q,\delta) : (|Q| > 1) \wedge (\exists n \in \mathbb{N}^*, \forall \mathcal{C} \in Q \ \ : ((G_\delta)^n(\mathcal{C}))_i = \mathcal{C}_{i-n})\}.$$
$$L_{SHIFT} = \{(Q,\delta) : (|Q| > 1) \wedge (\exists n \in \mathbb{N}^*, \forall \mathcal{C} \in Q \ \ : ((G_\delta)^n(\mathcal{C}))_i = \mathcal{C}_{i+n})\}.$$

Proof. Let $(Q,\delta) \in PER$. Let us consider any configuration $\mathcal{C}^*$ in which all the words over Q appear (it suffices to construct it as a suitable concatenation). Denoting the period of the cycle to which $\mathcal{C}^*$ belongs by n^*, it follows that:

$$\forall \overrightarrow{c} = (c_{-n^*}, \cdots, c_0, \cdots, c_{n^*}) \in Q^{2n^*+1} : \delta^{n^*}(\overrightarrow{c}) = c_0,$$

and therefore any other configuration $\mathcal{C} \in Q$ is n^*-periodic. For the R_{SHIFT} and the L_{SHIFT} classes the proof is exactly the same. $\square$

Let us introduce now the simplest periodic and shift-like CA: those having unitary length cycles. More precisely:

Definition 4 *Let the families of CA* (S_n, I_n)*,* (S_n, σ_n)*, and* (S_n, σ_n^{-1}) *be such that, for all* $n > 1$*:*

- $S_n = \{0, \cdots, n-1\}$.
- $\forall x, y, z \in S_n$:
  - · $I_n(x, y, z) = y$.
  - · $\sigma_n(x, y, z) = x$.
  - · $\sigma_n^{-1}(x, y, z) = z$.

It follows the same as for the nilpotency case. In fact, the proofs of the next two propositions are completely analogous to those of Proposition 5 and Proposition 6.

Proposition 7 *The equivalence classes represented by (S_2, I_2), (S_2, σ_2), and (S_2, σ_2^{-1}) correspond, respectively, to the families of periodic, right shift-like, and left shift-like CA. In other words, for any CA (Q, δ):*

$$(Q, \delta) \sim (S_2, I_2) \iff (Q, \delta) \in PER.$$

$$(Q, \delta) \sim (S_2, \sigma_2) \iff (Q, \delta) \in R_{SHIFT}.$$

$$(Q, \delta) \sim (S_2, \sigma_2^{-1}) \iff (Q, \delta) \in L_{SHIFT}.$$

Proposition 8 *The equivalence classes NIL, R_{SHIFT}, and L_{SHIFT} belong to the bottom of (CA*,$\leq$). In other words, for any non-singleton CA (Q, δ):*

$$(Q, \delta) \leq (S_2, I_2) \implies (S_2, I_2) \leq (Q, \delta).$$

$$(Q, \delta) \leq (S_2, \sigma_2) \implies (S_2, \sigma_2) \leq (Q, \delta).$$

$$(Q, \delta) \leq (S_2, \sigma_2^{-1}) \implies (S_2, \sigma_2^{-1}) \leq (Q, \delta).$$

Corollary 2 *The classes R_{SHIFT}, L_{SHIFT}, NIL, and PER are pairwise incomparable.*

Proof. Consider any non (spatially) periodic configuration and notice that its behavior could never be simultaneously of two types. □

5 Infinite bounded chains

Here we prove a non-trivial property concerning (CA*,$\leq$): the existence of two incomparable infinite chains having a common upper bound. This upper bound corresponds to the equivalence class represented by a "synchronization

CA" denoted by $(\mathcal{Q},\mathcal{D})$. More precisely, $(\mathcal{Q},\mathcal{D})$ is a suitable composition of the CA that solves a slightly modified version of the well-known firing-squad problem [10] with another one that simply transmits signals. Notice that it could be said that the order (CA*,$\leq$) "takes into account the algorithmical non-triviality of $(\mathcal{Q},\mathcal{D})$" because it admits (at least) a pair of infinite chains separating $(\mathcal{Q},\mathcal{D})$ from the minimum.

Let $\{(S_n,\eta_n)\}_{n>1}$ and $\{(S_n,\mu_n)\}_{n>1}$ be two CA families defined as follows:

- $S_n=\{0,\cdots,n-1\}$.
- $\eta_n(x,y,z)=\begin{cases} x & \text{if } x=y=z,\\ 0 & \text{otherwise.}\end{cases}$
- $\mu_n(x,y,z)=\min\{x,y,z\}$.

Before proving that the previous families are incomparable and infinite chains, notice that there exists a pair of points belonging to different families which are comparable. In fact, the initial points (S_2,η_2) and (S_2,μ_2) are isomorphic. On the other hand, they are located above the NIL class as it is proved in the next proposition.

Proposition 9 *For all $(Q_{NIL},\delta_{NIL})\in NIL$:*

$$\begin{aligned}(Q_{NIL},\delta_{NIL}) &< (S_2,\eta_2),\\ (Q_{NIL},\delta_{NIL}) &< (S_2,\mu_2).\end{aligned}$$

Proof. Let $(Q_{NIL},\delta_{NIL})\in NIL$. First $(S_2,\eta_2)\not\leq(Q_{NIL},\delta_{NIL})$ because (S_2,η_2) is not nilpotent (see Lemma 6). On the other hand, notice that $(S_2,0_2)\subseteq(S_2,\eta_2)^2$ because it suffices to consider $\varphi:S_2\to(S_2)^2$ such that $\varphi(x)=(0x)$. $\square$

Lemma 8 *Let $n>1$. For all $i\in\mathbb{N}^*$:*

$$\begin{aligned} n &= |\{\overrightarrow{x}\in(S_n)^i:(\eta_n)^i_{\mathcal{G}}(\overrightarrow{x},\overrightarrow{x},\overrightarrow{x})=\overrightarrow{x}\}|\\ &= |\{\overrightarrow{x}\in(S_n)^i:(\mu_n)^i_{\mathcal{G}}(\overrightarrow{x},\overrightarrow{x},\overrightarrow{x})=\overrightarrow{x}\}|.\end{aligned}$$

Proof. First, for all $x\in S_n$:

$$\begin{aligned}(\eta_n)^i_{\mathcal{G}}(x\cdots x,x\cdots x,x\cdots x) &= x\cdots x,\\ (\mu_n)^i_{\mathcal{G}}(x\cdots x,x\cdots x,x\cdots x) &= x\cdots x.\end{aligned}$$

Let $\vec{x} = (x_1 \cdots x_i) \in (S_n)^i$ be such that $\exists k \in \{1, \cdots, i\}$ with $x_k \neq x_{k+1}$. Without loss of generality, let us assume $x_k > x_{k+1}$. It follows:

$$((\eta_n)^i_{\mathcal{G}}(\vec{x}, \vec{x}, \vec{x}))_k = 0 \qquad \neq x_k,$$
$$((\mu_n)^i_{\mathcal{G}}(\vec{x}, \vec{x}, \vec{x}))_k \leq x_{k+1} < x_k. \qquad \square$$

Proposition 10 *For all $n > 1$:*

$$(S_n, \eta_n) < (S_{n+1}, \eta_{n+1}),$$
$$(S_n, \mu_n) < (S_{n+1}, \mu_{n+1}).$$

Proof. First $(S_n, \eta_n) \leq (S_{n+1}, \eta_{n+1})$ because $\eta_{n+1}|_{S_n} = \eta_n$. Let us suppose that there exist $i, j \in \mathbb{N}^*$ such that $(S_{n+1}, \eta_{n+1})^i \subseteq (S_n, \eta_n)^j$. Let $\varphi : (S_{n+1})^i \to (S_n)^j$ be a suitable injection. It follows that, if $\overrightarrow{x} \in (S_{n+1})^i$ is such that $(\eta_{n+1})^i_{\mathcal{G}}(\overrightarrow{x}, \overrightarrow{x}, \overrightarrow{x}) = \overrightarrow{x}$ then $(\eta_n)^j_{\mathcal{G}}(\varphi(\overrightarrow{x}), \varphi(\overrightarrow{x}), \varphi(\overrightarrow{x})) = \varphi(\overrightarrow{x})$ and we contradict Lemma 8. For $(S_n, \mu_n) < (S_{n+1}, \mu_{n+1})$ the argument is exactly the same. $\square$

Lemma 9 *Let $i, n \in \mathbb{N}^*$ with $n > 1$ and let $\vec{a} = (a_1 \cdots a_i) \in (S_n)^i$. It holds:*

$$(\eta_n)^i_{\mathcal{G}}(\vec{a}, \vec{a}, \vec{a}) = \begin{cases} (a_1 \cdots a_1) & \text{if } a_1 = a_2 = \cdots = a_i, \\ (0 \cdots 0) & \text{otherwise.} \end{cases}$$

$$(\mu_n)^i_{\mathcal{G}}(\vec{a}, \vec{a}, \vec{a}) = (a^* \cdots a^*), \ \text{where } a^* = \min\{a_1, \cdots, a_i\}.$$

Proof. It suffices to notice that, for all $\vec{x} = (x_{-i} \cdots x_0 \cdots x_i) \in (S_n)^{2i+1}$:

$$(\eta_n)^i(x_{-i} \cdots x_0 \cdots x_i) \neq 0 \iff x_{-i} = \cdots = x_0 = \cdots = x_i \neq 0,$$
$$(\mu_n)^i(x_{-i} \cdots x_0 \cdots x_i) = \min\{x_{-i}, \cdots, x_0, \cdots, x_i\}. \qquad \square$$

Proposition 11 *For all $n > 2, m > 2$: $(S_m, \mu_m) \not\leq (S_n, \eta_n)$.*

Proof. Let us suppose that there exist $i, j \in \mathbb{N}^*$ such that $(S_m, \mu_m)^i \subseteq (S_n, \eta_n)^j$ and let us denote by $\varphi : (S_m)^i \to (S_n)^j$ a suitable injection. It follows that:

$$\forall x \in S_m, \exists \varphi_x \in S_n \text{ such that } \varphi(x \cdots x) = (\varphi_x \cdots \varphi_x).$$

In fact,

$$\begin{aligned}\varphi(x\cdots x) &= \varphi((\mu_m)^i_{\mathcal{G}}(x\cdots x, x\cdots x, x\cdots x))\\ &= (\eta_n)^j_{\mathcal{G}}(\varphi(x\cdots x), \varphi(x\cdots x), \varphi(x\cdots x))\\ &= (\varphi_x\cdots\varphi_x) \text{ (by Lemma 9).}\end{aligned}$$

Let $x \in S_m$ be such that $0 \leq x < m-1$ and $\varphi(x\cdots x) \neq (0\cdots 0)$. It follows:

$$\begin{aligned}\varphi(x\cdots x) &= \varphi((\mu_m)^i_{\mathcal{G}}(m-1\cdots m-1, x\cdots x, m-1\cdots m-1))\\ &= (\eta_n)^j_{\mathcal{G}}(\varphi_{m-1}\cdots\varphi_{m-1}, \varphi_x\cdots\varphi_x, \varphi_{m-1}\cdots\varphi_{m-1})\\ &= (0\cdots 0).\end{aligned}$$

This is a contradiction. □

Proposition 12 For all $n > 2, m > 2$: $(S_n, \eta_n) \not\leq (S_m, \mu_m)$.

Proof. Let us suppose that there exist $i, j \in \mathbb{N}^*$ such that $(S_n, \eta_n)^i \subseteq (S_m, \mu_m)^j$ and let us denote by $\varphi : (S_n)^i \rightarrow (S_m)^j$ a suitable injection. As in the proof of Proposition 11, we can show that:

$$\forall x \in S_n, \exists \varphi_x \in S_m \text{ such that } \varphi(x\cdots x) = (\varphi_x\cdots\varphi_x).$$

Let $x, y \in S_n$ be such that $\varphi_x > \varphi_y > 0$ and $y \neq 0$. It follows:

$$\begin{aligned}\varphi(0\cdots 0) &= \varphi((\eta_n)^i_{\mathcal{G}}(x\cdots x, y\cdots y, x\cdots x))\\ &= (\mu_m)^j_{\mathcal{G}}(\varphi_x\cdots\varphi_x, \varphi_y\cdots\varphi_y, \varphi_x\cdots\varphi_x)\\ &= (\varphi_y\cdots\varphi_y)\\ &= \varphi(y\cdots y).\end{aligned}$$

This is a contradiction. □

In order to obtain an upper bound for the two previously introduced chains we are going to compose a pair of CA. One of them is related to the classical *firing-squad* problem introduced in [10] and which consists in designing a CA capable to synchronize "as soon as possible" an array of cells of arbitrary size. In Lemma 10 a result that appears in [9] concerning a slightly modified version of the original problem known as *two-ends firing-squad* is formally stated:

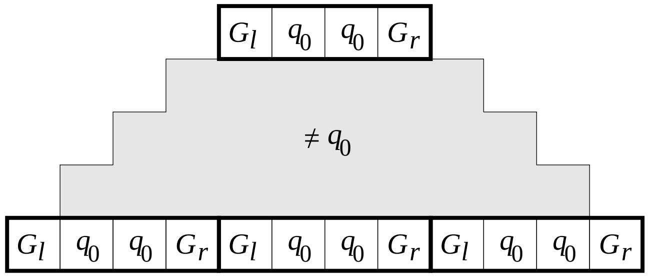

Fig. 2. Two-ends firing squad.

Lemma 10 *[9] There exists a CA (Q_{FS}, δ_{FS}) such that $\{G_l, G_r, q_0\} \subseteq Q_{FS}$ and which satisfies for all $n \in \mathbb{N}^*$ the following:*

- $(\delta_{FS})_G^{n+2}(G_l q_0 \cdots q_0 G_r, G_l q_0 \cdots q_0 G_r, G_l q_0 \cdots q_0 G_r) = (G_l q_0 \cdots q_0 G_r)$.
- *For all substring $\overrightarrow{x}$ of the triple concatenation*

$$(G_l q_0 \cdots q_0 G_r G_l q_0 \cdots q_0 G_r G_l q_0 \cdots q_0 G_r) \in (Q_{FS})^{3(n+2)}$$

such that $|\overrightarrow{x}| = 2k+1$ with $1 \leq k < (n+2)$, it holds:

$$(\delta_{FS})^k(\overrightarrow{x}) \neq q_0.$$

In Figure 2 appears schematically the case $n+2=4$. The CA to be composed with (Q_{FS}, δ_{FS}) is introduced in the next definition. Its cells simply transmit the signals (or states) coming from its left (resp. right) neighbor to its right (resp. left) neighbor keeping only its own information. More precisely,

Definition 5 *Let Q be an arbitrary set of states. The CA $(S_Q^{signal}, \delta_Q^{signal})$ is defined as follows:*

- $S_Q^{signal} = Q^3$*, and the states of S_Q^{signal} are denoted by $s = (s_l s_c s_r)$.*
- $\forall x, y, z \in S_Q^{signal} : \delta_Q^{signal}(x, y, z) = (x_l y_c z_r)$.

The composed CA is almost the "superposition" of the two previously introduced ones (with the set of signals $Q = \{0, 1\}$). The exception is done at the last step of the firing-squad period. More precisely, the only 1 signals not destroyed (transformed into 0) are those arriving simultaneously to a cell. Formally:

Definition 6 *Let $(\mathcal{Q}, \mathcal{D})$ be the CA such that $\mathcal{Q} = Q_{FS} \times S_{\{0,1\}}^{signal}$. We denote $\mathcal{D}((x_1, x_2), (y_1, y_2), (z_1, z_2))$ by $\mathcal{D}_{x,y,z}$. We define $\mathcal{D}_{x,y,z}$ as follows:*

$$\mathcal{D}_{x,y,z} = \begin{cases} (\delta_{FS}(x_1, y_1, z_1), 000) & \text{if } \begin{array}{l} \delta_{FS}(x_1, y_1, z_1) = q_0 \\ \text{and } \delta_{\{0,1\}}^{signal}(x_2, y_2, z_2) \neq (111). \end{array} \\ (\delta_{FS}(x_1, y_1, z_1), \delta_{\{0,1\}}^{signal}(x_2, y_2, z_2)) & \text{otherwise.} \end{cases}$$

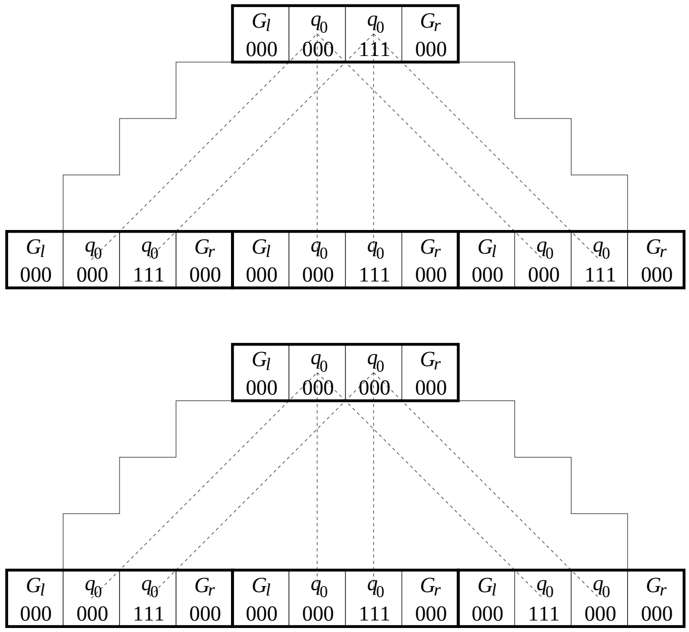

Fig. 3. "Simulating" by $(\mathcal{Q},\mathcal{D})^4$ the transitions of (S_3,η_3) for $(2,2,2)$ and $(2,2,1)$.

Proposition 13 *For all* $n>1$*:* $(S_n,\eta_n)<(\mathcal{Q},\mathcal{D})$.

Proof. Let $n\in\mathbb{N}^*$. $(S_n,\eta_n)\subseteq(\mathcal{Q},\mathcal{D})^{n+1}$ by the injection $\varphi: S_n\to\mathcal{Q}^{n+1}$ that follows:

$$\varphi(0)=((G_l,000),(q_0,000),\cdots,(q_0,000),(G_r,000)),$$
$$\varphi(x)=((G_l,000),\underbrace{(q_0,000),\cdots,(q_0,000),(q_0,111)}_{x\text{ with }0<x\leq n-1},(q_0,000),\cdots,(G_r,000)).$$

In fact, as it is shown in Figure 3,

$$\mathcal{D}_{\mathcal{G}}^{n+1}(\varphi(x),\varphi(y),\varphi(z))=\begin{cases}\varphi(x)\text{ if }\varphi(x)=\varphi(y)=\varphi(z),\\ \varphi(0)\text{ otherwise.}\end{cases}$$

$$=\varphi(\eta_n(x,y,z)).$$

On the other hand $(\mathcal{Q},\mathcal{D})\not\leq(S_n,\eta_n)$ because $(S_{n+1},\eta_{n+1})\not\leq(S_n,\eta_n)$. $\square$

Proposition 14 *For all $n > 1$: $(S_n, \mu_n) < (\mathcal{Q}, \mathcal{D})$.*

Proof. Let $n \in \mathbb{N}^*$. $(S_n, \eta_n) \subseteq (\mathcal{Q}, \mathcal{D})^{n+1}$ by the injection $\varphi : S_n \to \mathcal{Q}^{n+1}$ defined as follows:

$$\varphi(0) = ((G_l, 000), (q_0, 000), \cdots, (q_0, 000), (G_r, 000)),$$
$$\varphi(x) = (G_l, 000), \underbrace{(q_0, 111), \cdots, (q_0, 111)}_{x \text{ with } 0<x\leq n-1}, (q_0, 000), \cdots, (q_0, 000), (G_r, 000)).$$

In fact, as for η_n:

$$\mathcal{D}_{\mathcal{G}}^{n+1}(\varphi(x), \varphi(y), \varphi(z)) = \varphi(\min\{x, y, z\}) = \varphi(\mu_n\{x, y, z\}).$$

On the other hand $(\mathcal{Q}, \mathcal{D}) \not\leq (S_n, \mu_n)$ because $(S_{n+1}, \mu_{n+1}) \not\leq (S_n, \mu_n)$. $\square$

6 Infinite unbounded chains

In this section we prove that (CA*,$\leq$) has no maximum. Moreover, we prove that even maximal elements do not exist in (CA*,$\leq$). Therefore, for any CA X, the set of all the subautomata of all the powers of X will never cover all the CA classes. In other words, in a "grouping sense", there is no universal CA. As it is explained at the end of the section, this result gives us a lower-bound in the more general framework of "intrinsic-universality on CA" developed by Albert and Čulik II [1]. The proof is based on the existence of an unbounded infinite chain obtained after "processing" the next one.

Definition 7 *$\{(S_n, \Delta_n)\}_{n>1}$ is the family of CA such that, for each $n > 1$:*

- $S_n = \{0, \cdots, n-1\}$,
- $\Delta_n(x, y, z) = \begin{cases} x & \text{if } x = z, \\ y & \text{if } x \neq z. \end{cases}$

Notice that the first element of the previous family is located above the class NIL. In fact, if we consider the CA $(S_2, 0_2) \in NIL$ and (S_2, μ_2) introduced in previous chapters, where $0_2(x, y, z) = 0$ and $\mu_2(x, y, z) = \min\{x, y, z\}$, it follows:

Proposition 15 $(S_2, 0_2) < (S_2, \mu_2) < (S_2, \Delta_2)$.

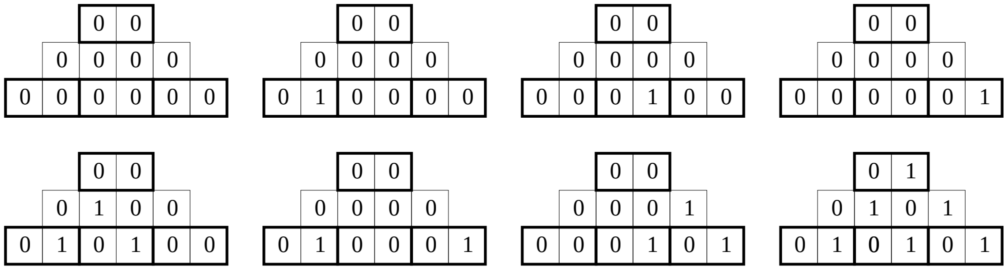

Fig. 4. Embedding (S_2, μ_2) into $(S_2, \Delta_2)^2$.

Proof. In Proposition 9 it was proved that $(S_2, 0_2) < (S_2, \mu_2)$. On the other hand, as it is shown in Figure 4, it is easy to check that $(S_2, \mu_2) \subseteq (S_2, \Delta_2)^2$ by the injection $\varphi : S_2 \to (S_2)^2$ such that $\varphi(x) = (0x)$. Finally, let us suppose that there exist $i, j \in \mathbb{N}^*$ such that $(S_2, \Delta_2)^i \subseteq (S_2, \mu_2)^j$. By Lemma 3, $(S_2, \Delta_2)^{2i} \subseteq (S_2, \mu_2)^{2j}$. By Lemma 8,

$$|\{\overrightarrow{x} \in (S_2)^{2j} : (\mu_2)^{2j}_{\mathcal{G}}(\overrightarrow{x}, \overrightarrow{x}, \overrightarrow{x}) = \overrightarrow{x}\}| = 2.$$

However,

$$|\{\overrightarrow{x} \in (S_2)^{2i} : (\Delta_2)^{2i}_{\mathcal{G}}(\overrightarrow{x}, \overrightarrow{x}, \overrightarrow{x}) = \overrightarrow{x}\}| \geq 4,$$

because for all $x, y \in S_2$:

$$(\Delta_2)^{2i}_{\mathcal{G}}(xy \cdots xy, xy \cdots xy, xy \cdots xy) = (xy \cdots xy). \qquad \square$$

The following is the key result of the present chapter. It says that "(S_n, Δ_n) is never a subautomaton of any power of any other CA with less states". Formally:

Proposition 16 *Let (Q, δ) be a CA and let $n \in \mathbb{N}^*$. It holds:*

$$|Q| < n \Longrightarrow \forall i \in \mathbb{N}^* : (S_n, \Delta_n) \not\subseteq (Q, \delta)^i.$$

Proof. Let (Q, δ) be a CA and let $n \in \mathbb{N}^*$ with $|Q| < n$. Suppose that there exists $i \in \mathbb{N}^*$ such that $(S_n, \Delta_n) \subseteq (Q, \delta)^i$. Then, by definition, there exists an injection $\varphi : S_n \to Q^i$ such that:

$$\forall x, y, z \in S_n : \varphi(\Delta_n(x, y, z)) = \delta^i_{\mathcal{G}}(\varphi(x), \varphi(y), \varphi(z)).$$

Let i_0 be the smallest index of Q^i for which there exist at least two elements of $\varphi(S_n)$ having different values (see Figure 5). Formally:

$$i_0 = \min\{k \in \mathbb{N}^* : \exists (x_1, \cdots, x_i), (y_1, \cdots, y_i) \in \varphi(S_n) \text{ such that } x_k \neq y_k\}.$$

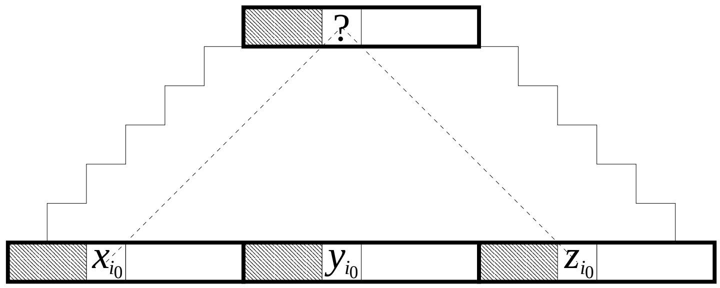

Fig. 5. Cell i_0 and the information it may access.

Notice that $i_0 \in \{1, \cdots i\}$ is well defined because $|S_n| > 1$. It follows:

$$\forall \overrightarrow{x}, \overrightarrow{z} \in \varphi(S_n) : \overrightarrow{x} \neq \overrightarrow{z} \Rightarrow x_{i_0} \neq z_{i_0}.$$

In fact, suppose that there exist $\overrightarrow{x}, \overrightarrow{z} \in \varphi(S_n)$ with $\overrightarrow{x} \neq \overrightarrow{z}$ such that $x_{i_0} = z_{i_0}$. By construction of i_0 there always exists $\overrightarrow{y} \in \varphi(S_n)$ such that $x_{i_0} \neq y_{i_0}$. This fact contradicts the following one:

$$\begin{aligned} x_{i_0} = (\delta^i_{\mathcal{G}}(\overrightarrow{x}, \overrightarrow{y}, \overrightarrow{x}))_{i_0} &= \delta^{2i+1}(x_{i_0}, \cdots, x_i, y_1 \cdots, y_i, x_1, \cdots, x_{i_0}) \\ &= \delta^{2i+1}(x_{i_0}, \cdots, x_i, y_1 \cdots, y_i, z_1, \cdots, z_{i_0}) \\ &= (\delta^i_{\mathcal{G}}(\overrightarrow{x}, \overrightarrow{y}, \overrightarrow{z}))_{i_0} = y_{i_0}. \end{aligned}$$

Finally it follows that $\beta : S_n \to Q$ with $\beta(x) = (\varphi(x))_{i_0}$ is an injection and therefore $n \leq |Q|$. This is a contradiction and therefore the proposition is concluded. □

In order to conclude that (CA*,≤) admits no maximum we need to prove a lemma that says that every CA is contained in all the powers of a suitable composition of itself with the CA that transmits signals introduced in Definition 5. More precisely,

Lemma 11 *For any CA (Q, δ) there exists a "normalized version" denoted by $(Q, \delta)^* = (Q^*, \delta^*)$ satisfying, for all $i \in \mathbb{N}^*$: $(Q, \delta) \subseteq (Q^*, \delta^*)^i$.*

Proof. Let us denote by B a state not belonging to any CA. Let (Q, δ) be a CA. We define $Q^* = S^{signal}_{\{Q \cup \{B\}\}}$ and for all $x, y, z \in Q^*$:

$$\delta^*(x, y, z) = \begin{cases} (\delta(x_l, y_c, z_r), \delta(x_l, y_c, z_r), \delta(x_l, y_c, z_r)) & \text{if } x_l, y_c, z_r \in Q, \\ \delta^{signal}_{\{Q \cup \{B\}\}}(x, y, z) & \text{otherwise.} \end{cases}$$

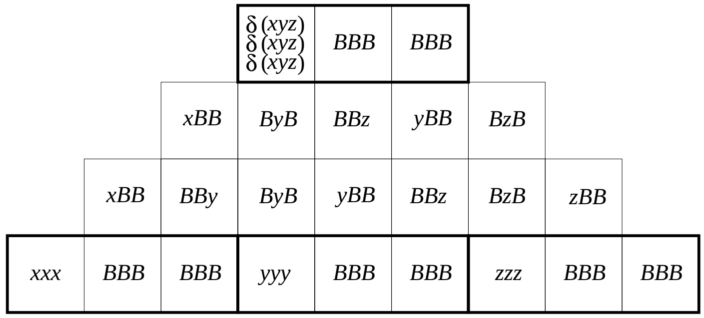

Fig. 6. Embedding (Q,δ) into $(Q^*,\delta^*)^i$ for $i=3$.

As it appears in Figure 6, for any $i \in \mathbb{N}^*$, $(Q,\delta) \subseteq (Q^*,\delta^*)^i$ by the injection $\varphi : Q \to (Q^*)^i$ defined as follows:

$$\varphi(x) = ((xxx),(BBB),\cdots(BBB)). \qquad \square$$

Proposition 17 *Let (Q,δ) be a CA and let $n \in \mathbb{N}^*$. It holds:*

$$|Q| < n \Longrightarrow (S_n^*,\Delta_n^*) \not\leq (Q,\delta).$$

Proof. Let (Q,δ) be a CA and let $n \in \mathbb{N}^*$ be such that $|Q| < n$. Suppose that there exist $i,j \in \mathbb{N}^*$ such that:

$$(S_n^*,\Delta_n^*)^i \subseteq (Q,\delta)^j.$$

By Lemma 11, $(S_n,\Delta_n) \subseteq (Q,\delta)^j$, and we contradict Proposition 16. $\square$

Corollary 3 *$(CA^*,\leq)$ has no maximum.*

Proposition 18 *For all $n \in \mathbb{N}^* : (S_n^*,\Delta_n^*) < (S_{(n+1)^3+1}^*,\Delta_{(n+1)^3+1}^*)$.*

Proof. First $(S_n^*,\Delta_n^*) \leq (S_{(n+1)^3+1}^*,\Delta_{(n+1)^3+1}^*)$ because $S_n^* \subseteq S_{(n+1)^3+1}^*$ and $\Delta_{(n+1)^3+1}^*\big|_{S_n^*} = \Delta_n^*$. Let us suppose that $(S_{(n+1)^3+1}^*,\Delta_{(n+1)^3+1}^*) \leq (S_n^*,\Delta_n^*)$. Considering that $|S_n^*| = (n+1)^3$, we contradict Proposition 17. $\square$

Proposition 19 *For every CA (Q,δ) there exists another one $(\tilde{Q},\tilde{\delta})$ such that $(Q,\delta) < (\tilde{Q},\tilde{\delta})$.*

Proof. It suffices to simply consider the superposition $(\tilde{Q},\tilde{\delta}) = (Q,\delta) \otimes (S_{|Q|+1}^*,\Delta_{|Q|+1}^*)$. By Proposition 4, $(Q,\delta) \leq (\tilde{Q},\tilde{\delta})$. On the other hand, if $(\tilde{Q},\tilde{\delta}) \leq (Q,\delta)$ then $(S_{|Q|+1}^*,\Delta_{|Q|+1}^*) \leq (Q,\delta)$, which is a contradiction. $\square$

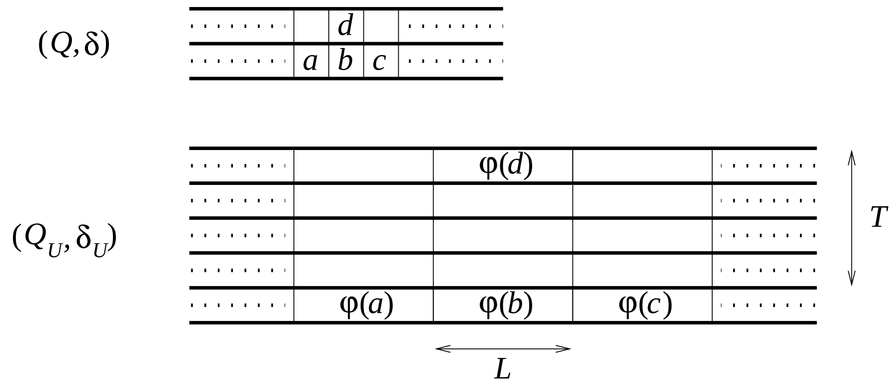

Fig. 7. The notion of intrinsic universality.

Corollary 4 *There are no maximal elements in* $(CA^*, \leq)$*.*

Remark 2 *The concept of intrinsic or self-referenced simulation on CA was introduced in [1]. It simply means that we "simulate directly a CA without passing through Turing machines". More precisely, as it is schematically shown in Figure 7, a CA* (Q_U, δ_U) *is said to be intrinsic-universal if any configuration of an arbitrary CA* (Q, δ) *can be encoded into a configuration of* (Q_U, δ_U) *so that each cell of the simulated CA is encoded as a block of cells of* (Q_U, δ_U) *of size* $L_{(Q,\delta)}$*. Each step of* (Q, δ) *is simulated by* $T_{(Q,\delta)}$ *steps of* (Q_U, δ_U)*. In [8] appears an intrinsic universal CA working in quasi-linear time but restricted to totalistic transitions (more precisely:* $T = O(|Q| \log(3|Q|)$ *and* $L = \log(3|Q|)$*). Our result means that, independently of the codification* $\varphi_{(Q,\delta)} : Q \to (Q_U)^L$*, we have* $T_{(Q,\delta)} > L_{(Q,\delta)}$ *(asymptotically).*

7 Concluding remarks

Throughout this work we have many times suggested the possibility of developing a complexity notion on CA based on the structure of $(\text{CA}^*, \leq)$. In fact, for any CA X, we have some evidence supporting the choice of the "longest" chain separating X from the minimum as a natural measure of its "complexity". In fact,

- The simplest CA analyzed here (nilpotents, shift-like, periodics) have all "minimal complexity".
- The only algorithmically non-trivial CA which appeared in this work (a modified version of the one that solves the firing-squad problem) has "infinite complexity".

Those CA separated from the minimum by an infinite chain could be classified in hierarchies by considering "the nature of the separating chain". Therefore, the first questions concerning the $(CA^*,\leq)$ structure arise:

- Is it a lattice?
- Does it admit dense chains?
- Does it admit ordinals?

A deeper understanding of the $(CA^*,\leq)$ structure should "start from the bottom". In other words, we should try to characterize those CA located immediately above the minimum and determine, for instance, whether all the "self-similar" CA belong to this category.

The non-existence of a maximum in $(CA^*,\leq)$ may be interpreted in the "intrinsic universality" framework as the impossibility of "simulating in real time". Improvements in our lower-bound could be done in the future. Also in the calculability domain, the following question seems very natural:

- In which part of $(CA^*,\leq)$ is located "the smallest" Turing-universal CA?

Finally, we would like to point out that our results can be easily generalized to CA of any dimension.

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