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Combinatorial theory of the Kontsevich–Zorich cocycle

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Abstract

In this work, three questions related to the Kontsevich–Zorich cocycle in the moduli space of quadratic differentials are studied by using combinatorial techniques.

The first two deal with the structure of the Rauzy–Veech groups of Abelian and quadratic differentials, respectively. These groups encode the homological action of almost-closed orbits of the Teichmüller geodesic flow in a given component of a stratum via the Kontsevich–Zorich cocycle. For Abelian differentials, we completely classify such groups, showing that they are explicit subgroups of symplectic groups that are commensurable to arithmetic lattices. For quadratic differentials, we show that they are also commensurable to arithmetic lattices of symplectic groups if certain conditions on the orders of the singularities are satisfied.

The third question deals with the realisability of certain algebraic groups as Zariski-closures of monodromy groups of square-tiled surfaces. Indeed, we show that some groups of the form $\mathrm{SO}^*(2d)$ are realisable as such Zariski-closures.

Keywords

Riemann surface, Quadratic differential, Abelian differential, Translation surface, Half-translation surface, Teichmüller dynamics, Kontsevich–Zorich cocycle, Rauzy–Veech algorithm, Square-tiled surface, Monodromy group

Théorie combinatoire du cocycle de Kontsevich–Zorich

Résumé

En ce travail, trois questions liées au cocycle de Kontsevich–Zorich dans l'espaces de modules des différentielles quadratiques sont étudies avec des techniques combinatoires.

Les deux premières impliquent la structure des groupes de Rauzy–Veech des différentielles abéliennes et quadratiques, respectivement. Ces groupes encodent l'action homologique des orbites presque fermées du flot géodésique de Teichmüller dans une composante connexe donnée d'une strate via le cocycle de Kontsevich–Zorich. Pour le cas abélien, on classifie complètement ces groupes et on montre qu'ils sont des sous-groupes explicites des groupes symplectiques, et qu'ils sont commensurables avec des réseaux arithmétiques. Pour le cas quadratique, on montre qu'ils sont aussi commensurables avec des réseaux arithmétiques si certaines conditions sur les ordres des singularités sont satisfaites.

La troisième question implique la réalisabilité de certain groupes algébriques comme adhérences de Zariski des groupes de monodromie des surfaces à petits carreaux. En fait, on montre que quelques groupes de la forme $\mathrm{SO}^*(2d)$ sont réalisables comme telles adhérences.

Mots-clés

Surface de Riemann, Différentielle quadratique, Différentielle abélienne, Surface de translation, Surface de demi-translation, Dynamique de Teichmüller, Cocycle de Kontsevich–Zorich, Algorithme de Rauzy–Veech, Surface à petits carreaux, Groupe de monodromie

Contents

Introduction

Ce travail s'inscrit dans le cadre de la dynamique de Teichmüller, un domaine riche dont les techniques combinent des méthodes provenant de l'analyse complexe, la géométrie différentielle, la géométrie algébrique et la combinatoire pour comprendre les propriétés dynamiques de l'espace de modules des surfaces de Riemann (ou, plus précisément, de son fibré cotangent). Après présenter quelques aspects basiques de la théorie dans le Chapitre 1, il se compose de trois résultats principaux : la classification complète des groupes de Rauzy–Veech des surfaces de translation (Chapitre 2), la classification partielle des groupes de Rauzy–Veech des surfaces de demi-translation et la simplicité des spectres de Lyapunov dans certains cas (Chapitre 3) et la réalisabilité de quelques groupes orthogonales quaternioniques comme adhérences de Zariski de certains groupes de monodromie des surfaces à petits carreaux (Chapitre 4).

Une *surface de Riemann* est une variété complexe et connexe de dimension complexe égal à 1. On considérera que des surfaces de Riemann compactes, possiblement avec un nombre fini de points enlevés. Les surfaces de Riemann sont, en particulier, des surfaces réelles de dimension égal a 2. Le théorème de classification des surfaces topologiques compactes implique alors que, au niveau topologique, une surface est complètement classifié par son *genre* (c'est-à-dire le nombre de « poignées ») et le nombre de points enlevés. Néanmoins, au niveau complexe il existent des surfaces différentes qui ont le même genre et nombre de points enlevés.

Le théorème d'uniformisation affirme que le revêtement universel de toute surface de Riemann de caractéristique d'Euler négative est le demi-plan complexe supérieur $\mathbb{H}$. Cet espace admet une structure complexe naturelle (héritée de la structure complexe de $\mathbb{C}$), ainsi qu'une métrique hyperbolique. De plus, les groupes $\mathrm{Aut}(\mathbb{H})$ des fonctions biholomorphes $\mathbb{H} \to \mathbb{H}$ et le groupe $\mathrm{Isom}^+(\mathbb{H})$ des isométries $\mathbb{H} \to \mathbb{H}$ qui préservent l'orientation coïncident : ils sont égales au groupe des *transformations de Möbius*. On obtient que toute surface de Riemann est de la forme $G\backslash\mathbb{H}$, où $G \leq \mathrm{Aut}(\mathbb{H}) = \mathrm{Isom}^+(\mathbb{H})$. En outre, on obtient l'équivalence entre les surfaces de Riemann et les surfaces hyperboliques. Cette équivalence peut être utilisée pour donner des exemples concrets des surfaces de Riemann.

On est prêt maintenant pour donner la définition de l'espace de Teichmüller des surfaces de Riemann. Si S est une surface topologique fixée, on dit que (X, f) est une *surface de Riemann marquée* si $f : S \to X$ est un homéomorphisme qui préserve l'orientation. Alors, l'*espace de Teichmüller* $\mathcal{T}(S)$ de S est l'espace de surfaces de Riemann marquées modulo les homéomorphismes de S qui sont isotopes à l'identité $S \to S$.

Pour définir une métrique dans l'espace de Teichmüller, on a besoin de la notion d'application quasiconforme. Un homéomorphisme $h: X \to Y$, où X et Y sont des surfaces de Riemann, est dit K-*quasiconforme*, pour un réel $K \geq 1$, s'il transforme des cercles infinitésimaux en ellipses infinitésimales dont les rapports d'aspect sont bornés par K. En plus, il est dit *quasiconforme* s'il est K-quasiconforme pour un $K \geq 1$. Pour une application $h \in$ Homéo(X, Y), on note $K_h = \inf\{K \geq 1 \mid h \text{ est } K\text{-quasiconforme}\}$ qui est égal a ∞ si h n'est pas quasiconforme. Alors, pour $[X, f], [Y, g] \in \mathcal{T}(S)$ on pose :

$$d_{\mathcal{T}}([X, f], [Y, g]) = \inf\left\{\frac{1}{2}\log K_h \;\middle|\; h \in \text{Homéo}(X, Y) \text{ isotope à } gf^{-1}\right\}.$$

D'après le théorème d'application de Riemann mesurable, l'espace cotangent $T^*_X\mathcal{T}(S)$ peut être identifié avec l'espace $\mathbb{Q}_X$ des différentielles quadratiques méromorphes définies sur X avec des pôles d'ordre au plus 1. On appelle le fibré cotangent de $\mathcal{T}(S)$ l'*espace de Teichmüller des différentielles quadratiques*.

Les théorèmes de Teichmüller, d'autre part, affirment que pour toutes $[X, f], [Y, g] \in \mathcal{T}(S)$ il existe une application quasiconforme $\sigma: X \to Y$ dans la classe d'isotopie de gf^{-1} avec $K_\sigma = d_{\mathcal{T}}([X, f], [Y, g])$. Cette application est appelée une *application extrémale*. En outre, ils affirment qu'il existent des atlas sur X et Y qui sont compatibles avec les structures complexes et dont les applications de transition sont de la forme $\pm z + c$, sauf pour voisinages d'un nombre fini de points, tels que σ est une dilatation par $\sqrt{K_\sigma}$ dans la direction horizontal et une contraction par $\sqrt{1/K_\sigma}$ dans la direction vertical. Un tel atlas sur X est équivalent à une différentielle quadratique méromorphe sur X avec des pôles au plus simples. Alors, les applications extrémales peuvent être interprétées comme des matrices de la forme diag(e^t, e^{-t}) en termes de différentielles quadratiques sur les surfaces de Riemann. L'action de ce groupe est appelée le *flot géodésique de Teichmüller* sur l'espace de Teichmüller des différentielles quadratiques. On peut aussi définir, d'une manière analogue, une action de $\mathrm{SL}(2, \mathbb{R})$ sur l'espace de Teichmüller des différentielles quadratiques. Alors, le flot géodésique de Teichmüller correspond à la sous-action par le sous-groupe des matrices diagonales.

Le groupe modulaire Mod(S) agit sur $\mathcal{T}(S)$ en changeant le marquage. L'*espace de modules* des surfaces de Riemann $\mathcal{M}(S)$ est le quotient de $\mathcal{T}(S)$ par l'action de Mod(S). Ainsi, il peut être interprété comme l'espace des surfaces de Riemann non-marquées. Le fibré cotangent de $\mathcal{M}(S)$ est appelé l'*espace de modules des différentielles quadratiques*. L'action de $\mathrm{SL}(2, \mathbb{R})$ est compatible avec l'action de Mod(S), donc elle est aussi définie sur l'espace de modules des différentielles quadratiques.

Une *surface plate* est une surface de Riemann munie d'une différentielle quadratique méromorphe avec des pôles au plus simples. On peut définir les espaces de Teichmüller et de modules des surfaces plates : ils sont exactement les espaces de Teichmüller et de modules des différentielles quadratiques, respectivement. En plus, certaines différentielles quadratiques peuvent être écrites comme des carrés globaux des différentielles abéliennes. Dans ce cas, la surface plate est appelée une *surface de translation*. Elle est appelé une *surface de demi-translation* sinon.

Une manière combinatoire et équivalente de définir une surface plate est la suivante : une surface plate est une collection finie de polygones sur $\mathbb{C}$ avec une manière d'identifier paires de côtés « opposés » par translations et rotations d'angle π. Ces collections sont identifiées par une relation d'équivalence : deux telles collections sont équivalentes s'il est possible de couper et coller la première collection le long de lignes droites et de recoller les morceaux en utilisant les identifications pour obtenir la deuxième collection. En d'autres termes, il existe une série d'*opérations de coupure et recollement* qui transforme l'une en l'autre.

Le *cocycle de Kontsevich–Zorich* est le cocycle qui code l'action homologique du flot géodésique de Teichmüller ou de l'action de $\mathrm{SL}(2, \mathbb{R})$. L'algorithme de Rauzy–Veech est une manière combinatoire de comprendre le cocycle de Kontsevich–Zorich : il nous donne des bases des groupes d'homologie et des opérations de coupure et recollement concrètes qui permettent de calculer des matrices explicites. Ces matrices sont symplectiques car elles préservent les nombres d'intersection des courbes. Les *groupes de Rauzy–Veech* sont les sous-groupes des groupes symplectiques induits par cet algorithme. Dans le Chapitre 2, on calcule ces groupes pour le cas abélienne :

Théorème A. *Les groupes de Rauzy–Veech des surfaces de translation sont des sous-groupes d'indice fini explicites des groupes symplectiques.*

Dans le Chapitre 3, on calcule ces groupes pour certains cas quadratiques :

Théorème B. *Les groupes de Rauzy–Veech de certaines surfaces de demi-translation sont des sous-groupes d'indice fini explicites des groupes symplectiques. Alors, dans ces cas, les spectres de Lyapunov du cocycle de Kontsevich–Zorich sont simples.*

Une *surface à petits carreaux* est une surface de translation composé des carrés unitaires dont les côtés sont identifiés par translation. L'action de $\mathrm{SL}(2, \mathbb{Z})$ préserve les surfaces à petits carreaux dites *réduites*. Le sous-groupe de $\mathrm{SL}(2, \mathbb{Z})$ qui préserve une surface à petits carreaux fixée M est appelé son *groupe de Veech* $\mathrm{SL}(M)$ et il est un sous-groupe d'indice fini de $\mathrm{SL}(2, \mathbb{Z})$. Le *groupe de monodromie* de M est le sous-groupe du groupe symplectique induit par l'action en homologie de $\mathrm{SL}(M)$. La liste des groupes algébriques qui sont réalisables comme adhérences de Zariski des groupes de monodromie des surfaces à petits carreaux est contrainte par des conditions algébro-géométriques. Néanmoins, on ne sais pas quels groupes de cette liste sont en effet réalisables. Dans le Chapitre 4, on montre que certains groupes le sont :

Théorème C. *Certains groupes de la forme* $\mathrm{SO}^*(2d)$*, dits groupes orthogonales quaternioniques, sont réalisables comme groupes de monodromie des surfaces à petits carreaux.*

Chapter 1

Background

In this chapter, we present the required background on Teichmüller geometry and dynamics. Parts of the following exposition were adapted from the minicourses given by Christian Leininger and Carlos Matheus during the summer school "Teichmüller dynamics, mapping class groups and applications" that took place at Institut Fourier in June 2018.

In an attempt to be fair when attributing credit, we will refer to the original articles if possible even when they are very old or difficult to read. Nevertheless, there are excellent books and surveys containing most of what is covered in this chapter. A fairly incomplete list is the following: the books by Farb and Margalit [FM12], Imayoshi and Taniguchi [IT92], Ahlfors [Ahl06], Hubbard [Hub06; Hub16] and Labourie [Lab13], the surveys by Forni and Matheus [FM14], Wright [Wri15], Zorich [Zor06] and Viana [Via06] and the lecture notes by Yoccoz [Yoc10].

1.1 Teichmüller geometry

Teichmüller geometry is the study of collections of Riemann surfaces. It is one of many examples in mathematics that show that studying suitable families of objects may provide valuable information about the individual objects themselves. The main actors are the Teichmüller space and the moduli space of Riemann surfaces. The former is the collection of Riemann surfaces up to diffeomorphisms that are isotopic to the identity and it is simply connected (in fact, homeomorphic to a ball in a Euclidean space!). The latter is the quotient of the former by the mapping class group and its topology and geometry are quite mysterious.

We will start this chapter with an overview of results relating complex and hyperbolic structures on surfaces. These results are already very deep and insightful. Many of them took quite a long time to be rigorously proved.

Then, we will define the Teichmüller space of Riemann surfaces, and the Teichmüller metric and the Teichmüller geodesic flow on it. To this end, we will develop some of the theory of quasiconformal mappings, which are homeomorphisms mapping infinitesimal circles to infinitesimal ellipses of bounded aspect ratios.

Afterwards, we will discuss the topic of deformations of Riemann surfaces via Beltrami and quadratic differentials. These spaces can be interpreted as the tangent and cotangent bundles of the Teichmüller space of Riemann surfaces, respectively. The main results of this part are the measurable Riemann mapping theorem, due to Alhfors and Bers, and one of Teichmüller's theorems.

We finish this section by discussing the action of the mapping class group of the Teichmüller space of Riemann surfaces. The quotient by this action is the moduli space of Riemann surfaces, which is the setting in which our work is actually developed.

1.1.1 Riemann surfaces and the uniformization theorem

A *Riemann surface* is a manifold (possibly with a nonempty boundary) modelled over $\mathbb{C}$ whose transition maps are holomorphic. That is, it is a one-dimensional complex-analytic manifold. An atlas defining such a manifold is called a *complex structure*. Since it is always possible to restrict to connected components, every Riemann surface that we consider will be connected.

We say that a map between two Riemann surfaces is *holomorphic* if it is holomorphic when composed with any of the relevant coordinate charts. If this map is bijective and its inverse is also holomorphic, we say that it is *biholomorphic*. When a biholomorphic map exists between two Riemann surfaces, we think of them as being "equal up to renaming" and we say that such surfaces are *biholomorphic*. Thus, in more abstract terms, biholomorphic maps are the isomorphisms in the category of Riemann surfaces.

A surface is said to be *closed* if it is compact and has an empty boundary. Since complex-analytic manifolds are orientable when considered as real manifolds, the underlying topological surface of a Riemann surface is completely characterised by an integer $g \geq 0$, which is called the *genus* of the surface [Dyc88; Ale15; Bra21]. Intuitively, this number is interpreted as the number of "holes": a genus-zero surface is a sphere, a genus-one surface is a torus and so on. While the topological picture is quite simple, the complex-analytic picture is much more complicated. Indeed, knowing that two Riemann surface have the same genus is far from enough to conclude that they are biholomorphic.

We will allow punctures on closed Riemann surfaces, that is, the removal of finitely many points. In this case, we define the genus to be the same as it was before removing such points. The *Euler characteristic* of a punctured closed Riemann surface X is defined to be the integer $\chi(X) = 2 - 2g - n$, where n is the number of punctures.

Given a Riemann surface X, we will write $\widetilde{X}$ for its universal cover. The space $\widetilde{X}$ is a surface and admits a complex structure such that the covering map $q\colon \widetilde{X} \to X$ is locally biholomorphic. Indeed, this structure is defined by pulling the coordinate charts of X back by q.

Let $\mathbb{H}$ be the upper half-plane $\{z \in \mathbb{C} \mid \operatorname{Im}(z) > 0\}$. This space admits a natural Riemann surface structure inherited from the complex structure of $\mathbb{C}$. The uniformization theorem [Poi08; Koe09; Koe10b; Koe10a; Koe11; Koe12; Koe14] implies the following:

Theorem 1.1.1 (Uniformization theorem). *Let X be a Riemann surface with $\chi(X) < 0$ and let $\widetilde{X}$ be its universal cover. Then, exists a biholomorphic map $D\colon \widetilde{X} \to \mathbb{H}$.*

The group of deck transformations of the covering map $q\colon \widetilde{X} \to X$ is naturally isomorphic to $\pi_1(X)$. We will identify these two groups by defining a group action of $\pi_1(X)$ on $\widetilde{X}$ by biholomorphic deck transformations.

Let $\mathrm{Aut}(\mathbb{H})$ be the group of biholomorphic maps $\mathbb{H} \to \mathbb{H}$. It is well-known that a map $T\colon \mathbb{H} \to \mathbb{H}$ belongs to $\mathrm{Aut}(\mathbb{H})$ if and only if it is a *Möbius transformation*, that is, $T(z) = \frac{az+b}{cz+d}$ for $a, b, c, d \in \mathbb{R}$ satisfying $ad - bc = 1$. This fact allows us to define a group action of $\mathrm{SL}(2, \mathbb{R})$ on $\mathbb{H}$. The kernel of this action is the subgroup $\{\mathrm{Id}, -\mathrm{Id}\}$, which shows that $\mathrm{Aut}(\mathbb{H})$ can be identified with $\mathrm{SL}(2, \mathbb{R})/\{\mathrm{Id}, -\mathrm{Id}\} = \mathrm{PSL}(2, \mathbb{R})$.

Let X be a Riemann surface with $\chi(X) < 0$ and let D be as in Theorem 1.1.1. We have that, for every $\gamma \in \pi_1(X)$, the map $\rho(\gamma) = D\gamma D^{-1}\colon \mathbb{H} \to \mathbb{H}$ is biholomorphic, so we obtain a well-defined group monomorphism $\rho\colon \pi_1(X) \to \mathrm{Aut}(\mathbb{H})$. The group $G = \rho(\pi_1(X))$ is a subgroup of $\mathrm{PSL}(2, \mathbb{R})$ which is uniquely defined up to conjugation. We can define a Riemann surface structure on $G\backslash\mathbb{H}$ by pushing the complex structure of $\mathbb{H}$ forward by the covering map p. By letting $q^{-1}\colon X \to \widetilde{X}$ be any map such that $q^{-1}q = \mathrm{Id}$, we define a map $\sigma = pDq^{-1}\colon X \to G\backslash\mathbb{H}$. This map does not depend on the choice of q^{-1}. Indeed, if $\tilde{x}, \tilde{y} \in \widetilde{X}$ satisfy $q(\tilde{x}) = q(\tilde{y})$, then there exists $\gamma \in \pi_1(X)$ such that $\gamma \cdot \tilde{x} = \tilde{y}$. Therefore, $\rho(\gamma) \cdot D(\tilde{x}) = D(\tilde{y})$ and we conclude that $pD(\tilde{x}) = pD(\tilde{y})$. Furthermore, the map $\sigma\colon X \to G\backslash\mathbb{H}$ is biholomorphic since both p and q are locally biholomorphic. We have, thus, proved the following corollary of Theorem 1.1.1:

Corollary 1.1.2. *Let X be a Riemann surface with $\chi(X) < 0$. Then, there exists a subgroup G of* $\mathrm{PSL}(2, \mathbb{R})$ *such that X and $G\backslash\mathbb{H}$ are biholomorphic. Moreover, the group G is unique up to conjugation.*

The previous theorem and corollary allow us to think of the universal cover of any Riemann surface of negative Euler characteristic as the upper half-plane $\mathbb{H}$ and of the Riemann surface itself as the quotient of the of $\mathbb{H}$ by some group G. The subgroups of $\mathrm{PSL}(2, \mathbb{R})$ arising in this way are called *Fuchsian groups* and, while they are rich and interesting objects, their study is beyond the scope of this thesis.

1.1.2 Hyperbolic surfaces and Riemann surfaces

The upper half-plane carries a natural Riemannian metric given by $ds = \frac{|dz|}{\mathrm{Im}(z)}$. That is, the length of a curve $\theta\colon [0, 1] \to \mathbb{H}$ is given by $\ell_{\mathbb{H}}(\theta) = \int_0^1 \frac{|\theta'(t)|}{\mathrm{Im}(\theta(t))} dt$. This metric is *hyperbolic* in the sense that its curvature is constant and equal to -1. The Riemannian manifold $(\mathbb{H}, ds)$ is called the *hyperbolic plane*. We will usually refer to this space as $\mathbb{H}$ for simplicity, as we will not endow $\mathbb{H}$ with any other Riemannian metric. The group $\mathrm{Isom}^+(\mathbb{H})$ of orientation-preserving isometries of $\mathbb{H}$ is exactly $\mathrm{PSL}(2, \mathbb{R})$ acting by Möbius transformations. Moreover, the geodesics of $\mathbb{H}$ are exactly the circles and lines orthogonal to the horizontal axis $\mathbb{R}$.

The nontrivial elements of the group $\mathrm{Isom}^+(\mathbb{H})$ are classified as follows:

Proposition 1.1.3. *Any element $T \in \mathrm{PSL}(2, \mathbb{R}) \setminus \{\mathrm{Id}\}$ satisfies exactly one of the following:*

1. *It fixes a point in $\mathbb{H}$ and is conjugate to $\frac{\cos(\theta)z-\sin(\theta)}{\sin(\theta)z+\cos(\theta)}$ for some $\theta \in \mathbb{R}$. In this case, T is said to be elliptic.*
2. *It fixes two points in $\mathbb{R} \cup \{\infty\}$, it acts by translation on a geodesic and is conjugate to $\lambda^2 z$ for $\lambda \in \mathbb{R}$. In this case, T is said to be hyperbolic.*
3. *It fixes a unique point in $\mathbb{R} \cup \{\infty\}$ and is conjugate to $z + 1$. In this case, T is said to be parabolic.*

A *hyperbolic surface* is a two-dimensional manifold modelled over $\mathbb{R}^2$ endowed with a hyperbolic Riemannian metric. Given a closed Riemann surface $X = {}_G\backslash^{\mathbb{H}}$ with $\chi(X) < 0$, we can push the hyperbolic metric of $\mathbb{H}$ forward to X to define a hyperbolic metric on it. By definition, the covering map $p\colon \mathbb{H} \to X$ is a local isometry. We will always endow such a Riemann surface with this metric and we will denote $\ell_X(\theta)$ for the induced length of a curve θ.

Conversely, if X is a surface endowed with metric locally isometric to $\mathbb{H}$, the Killing–Hopf theorem [Kil91; Hop26] implies that the universal cover of X is isometric to $\mathbb{H}$. As for the case of Riemann surfaces, we can then prove that X is isometric to ${}_G\backslash^{\mathbb{H}}$ for some subgroup $G \leq \mathrm{PSL}(2, \mathbb{R})$.

The following proposition completes the parallel between Riemann and hyperbolic surfaces, showing that the hyperbolic and complex structures are compactible.

Proposition 1.1.4. *Let X, Y be closed Riemann surfaces of genus at least two and $f\colon X \to Y$ be an orientation-preserving homeomorphism. Then, f is biholomorphic if and only if it is an isometry.*

Proof. In both cases, f lifts to a map $\tilde{f}\colon \mathbb{H} \to \mathbb{H}$. The proof then follows from the equality $\mathrm{Aut}(\mathbb{H}) = \mathrm{Isom}^+(\mathbb{H})$. □

Hyperbolic geodesics

Simple hyperbolic geodesics on hyperbolic surfaces can be obtained by "straightening out" any simple closed curve on the surface, as shown by the following proposition. This is a very useful fact that, in particular, allows us to construct Riemann surfaces from hyperbolic models below.

Proposition 1.1.5. *Let X be a closed Riemann surface with $\chi(X) < 0$. Let $\theta\colon [0, 1] \to X$ be a non-null-homotopic closed curve and $\gamma = [\theta] \in \pi_1(X)$. Then,*

1. *there exists a unique closed geodesic θ^* such that θ and θ^* are freely homotopic;*
2. *$\rho(\gamma)$ is hyperbolic; and*
3. *$\ell_X(\theta^*) = 2\cosh^{-1}\left(\frac{|\operatorname{tr}\rho(\gamma)|}{2}\right)$.*

Moreover, θ^* is simple if θ is simple.

Proof. Let $x = \theta(0) = \theta(1)$. For any $\tilde{x} \in \mathbb{H}$ such that $p(\tilde{x}) = x$, there exists a lift $\tilde{\theta}_{\tilde{x}}\colon [0, 1] \to \mathbb{H}$ satisfying $\tilde{\theta}_{\tilde{x}}(0) = \tilde{x}$. Fix $\tilde{x} \in p^{-1}(x)$ and let $\tilde{\theta}\colon \mathbb{R} \to \mathbb{H}$ be the curve such that $\tilde{\theta}|_{[n,n+1]} = \tilde{\theta}_{\gamma^n\cdot\tilde{x}}$ for each integer n. We have that γ fixes the two distinct points of $\lim_{n\to\infty}\gamma^n \cdot \tilde{x}$ and $\lim_{n\to-\infty}\gamma^n \cdot \tilde{x}$ of $\mathbb{R} \cup \{\infty\}$ and is, thus, hyperbolic. Therefore, it fixes the geodesic $\tilde{\theta}^*$ having these two points as endpoints. We can define θ^* as the projection of $\tilde{\theta}^*$ on X. To prove uniqueness, it is sufficient to show that any other geodesic segment freely homotopic to θ has the same endpoints as $\tilde{\theta}^*$. This is a consequence of the fact that a free homotopy between two curves implies that they are at a bounded distance by compactness of $[0, 1] \times [0, 1]$. The formula for the length follows from an elementary computation.

Finally, assume that θ is simple. Therefore, $\tilde{\theta}$ is also simple, and so is $\tilde{\theta}^*$ as geodesics of $\mathbb{H}$ are always simple. Furthermore, $\gamma = [\theta] = [\theta^*] \in \pi_1(X)$ is primitive, so θ^* is not a multiple of a curve. □

Explicit construction of Riemann surfaces

The following construction is an example of the fruitful relation between hyperbolic geometry and the theory of Riemann surfaces. Indeed, it is possible to construct all punctured closed Riemann surfaces with negative Euler characteristic using hyperbolic geometry. We will sketch this construction instead of providing rigorous proofs.

Step 1. Right-angled hexagons of $\mathbb{H}$ having geodesic boundaries are determined (up to isometry) by alternating side lengths. We allow the vertices of these hexagons to lie in $\mathbb{R} \cup \{\infty\}$. Gluing two of these hexagons along their geodesic boundaries produces a "pair of pants" (a sphere with three "holes", which can be punctures or boundary components) with geodesic boundaries.

Step 2. Pair of pants in $\mathbb{H}$ with geodesic boundaries are determined (up to isometry) by the lengths of their boundaries.

Step 3. Gluing hyperbolic pairs of pants along their geodesic boundaries, in such a way that no boundary component remains, produces a hyperbolic surface and, therefore, a Riemann surface. If some vertices of the original hexagons belonged to $\mathbb{R} \cup \{\infty\}$, the resulting surface has punctures.

Step 4. Every punctured closed Riemann surface with negative Euler characteristic can be built in this way. Indeed, given such a surface X we can find $3g - 3 + n$ disjoint closed simple curves, where $g \geq 0$ is the genus of X, inducing a pair of pants decomposition. These curves can be straightened out to simple closed geodesic by the previous proposition, so we obtain a pair of pants decomposition with geodesic boundaries.

Remark 1.1.6. This construction is closely related to the *Fenchel–Nielsen coordinates* of the Teichmüller space of Riemann surfaces, explained in the next section.

1.1.3 The Teichmüller and moduli spaces of Riemann surfaces

Let S be an orientable topological (possibly punctured) closed surface. A *marked Riemann surface* is a pair (X, f), where X is a Riemann surface whose underlying topological surface is S and $f : S \to X$ is an orientation-preserving homeomorphism called a *marking*. We say that two marked Riemann surface (X, f) and (Y, g) are equivalent if there exists a biholomorphic map

$\sigma: X \to Y$ such that σf and g are isotopic. We define the *Teichmüller space* of S to be the set of marked Riemann surfaces up to equivalence:

$$\mathcal{T}(S) = \{[X, f] \mid (X, f) \text{ is a marked Riemann surface}\}.$$

This space was originally defined by Teichmüller [Tei44].

The Teichmüller metric

The Teichmüller space carries a natural metric called the *Teichmüller metric*. In order to define it, we first need to define quasiconformal diffeomorphisms.

Let $U, V \subseteq \mathbb{C}$ and $f: U \to V$ be an orientation-preserving diffeomorphism. Given $K \geq 1$, we say that f is K-*quasiconformal* if one has that

$$\frac{\|df_z\|^2}{\det(df_z)} \leq K \text{ for every } z \in U.$$

In the previous expression, $\|df_z\| = \sup_{v \neq 0} \frac{|df_z(v)|}{|v|}$ is the usual norm of linear functions.

Intuitively, a diffeomorphism is K-quasiconformal if it sends infinitesimal circles to infinitesimal ellipses whose aspect ratios are bounded by K.

Given a map $f: U \to V$, we define its *coefficient of quasiconformality* $K_f \in \mathbb{R} \cup \{\infty\}$ by

$$K_f = \inf\{K \geq 1 \mid f \text{ is } K\text{-quasiconformal}\}, \text{ with the usual convention } \inf \varnothing = \infty.$$

We say that f is *quasiconformal* if $K_f < \infty$. This notion was introduced by Grötzsch [Grö28a; Grö28b].

If f is holomorphic, its coefficient of quasiconformality is 1. Moreover, pre- or post-composing a diffeomorphism with a holomorphic map does not change its coefficient of quasiconformality.

Now, let $f: X \to Y$ be an orientation-preserving diffeomorphism between two Riemann surfaces X and Y. We say that f is *quasiconformal* if it is quasiconformal in local coordinates. The coefficient of quasiconformality is well-defined, as the transition maps of X and Y are holomorphic.

As expected, the coefficient of quasiconformality behaves well under compositions and inverses:

Proposition 1.1.7. *Let X, Y and Z be Riemann surfaces and $f: X \to Y$, $g: Y \to Z$ be K- and K'-quasiconformal diffeomorphisms, respectively. Then, f^{-1} is also K-quasiconformal and gf is KK'-quasiconformal.*

The Teichmüller metric $d_{\mathcal{T}}$ on $\mathcal{T}(S)$ is defined as

$$d_{\mathcal{T}}([X, f], [Y, g]) = \inf\left\{\frac{1}{2}\log K_h \;\middle|\; h \in \mathrm{Diffeo}^+(X, Y) \text{ isotopic to } gf^{-1}\right\}.$$

This metric is not a Riemannian metric, but a Finsler metric. Indeed, it is induced by a weak norm defined on the tangent space at each point of $\mathcal{T}(S)$. The resulting topology makes the Teichmüller space homeomorphic to $\mathbb{R}^{6g-6+2n}$, where $g \geq 0$ is the genus and n is the number of punctures of S. This can be seen in terms of the explicit construction of Riemann surfaces in terms of hyperbolic models. Indeed, the parameters are $3g-3+n$ positive real numbers representing the lengths of the boundary components of the pairs of pants (which are called the *length parameters*), together with $3g-3+n$ real numbers representing possible rotations along the boundaries when gluing such pairs of pants (which are called the *twist parameters*). These coordinates for the Teichmüller are called Fenchel–Nielsen coordinates and are real-analytic [FN03].

From now on, we will assume that $3g-3+n > 0$, so the Teichmüller space is not zero-dimensional. This condition is stronger than $\chi(S) < 0$. Moreover, the only surface with negative Euler characteristic and $3g-3+n \leq 0$ is the thrice-punctured sphere.

Quasiconformal homeomorphisms

Let X and Y be Riemann surfaces. If we endow the group $\mathrm{Homeo}(X,Y)$ of homeomorphisms between X and Y with the compact-open topology, then the closure of the subset of K-quasiconformal diffeomorphisms is compact. We say that a homeomorphism $f\colon X \to Y$ is *K-quasiconformal* if it belongs to this set. As for diffeomorphisms, given $f \in \mathrm{Homeo}(X,Y)$ we define its *coefficient of quasiconformality* by

$$K_f = \inf\{K \geq 1 \mid f \text{ is } K\text{-quasiconformal}\}$$

and we say that f is *quasiconformal* if $K_f < \infty$.

While this definition is useful, it is difficult to decide if a map is quasiconformal by relying on it alone. Therefore, we will now give an equivalent and more concrete definition of quasiconformality.

Let $U, V \subseteq \mathbb{C}$ and $f\colon U \to V$ be an orientation-preserving homeomorphism. Assume that f has weak derivatives $\frac{\partial f}{\partial x}$ and $\frac{\partial f}{\partial y}$ which are represented by locally integrable functions. Let

$$f_z = \partial_z f = \frac{1}{2}\left(\frac{\partial f}{\partial x} - i\frac{\partial f}{\partial y}\right), \quad f_{\bar z} = \partial_{\bar z} f = \frac{1}{2}\left(\frac{\partial f}{\partial x} + i\frac{\partial f}{\partial y}\right).$$

Then, f is K-quasiconformal if and only if $|f_{\bar z}| \leq \kappa |f_z|$ almost everywhere, where $K = \frac{1+\kappa}{1-\kappa}$. Such a map is actually differentiable almost everywhere and we have that $\frac{\|df\|^2}{\det dt} \leq K$ almost everywhere. Moreover, orientation-preserving homeomorphisms that are diffeomorphisms and satisfy $\frac{\|df\|^2}{\det dt} \leq K$ in all but finitely many points are K-quasiconformal. Finally, it is clear that these definitions can be extended to Riemann surfaces via local coordinates.

The next theorem by Grötzsch completely characterises quasiconformal maps in a simple case, which is useful to develop intuition on how they behave.

Theorem 1.1.8. *Given $a, b \geq 0$, let $R_a = [0, a] \times [0, 1] \subseteq \mathbb{C}$ and $R_b = [0, b] \times [0, 1] \subseteq \mathbb{C}$. If $f : R_a \to R_b$ is an orientation-preserving K-quasiconformal homeomorphism, then $K \geq \frac{b}{a}$. Moreover, equality is attained if and only if $f(x + iy) = \frac{b}{a}x + iy$ for every $x \in [0, a]$ and $y \in [0, 1]$.*

Proof. Since f is K-quasiconformal, the inequality $|f_x(p)|^2 \leq \|df_p\|^2 \leq K \det df$ holds almost everywhere. Therefore,

$$b^2 \leq \left(\int_0^1 \int_0^a |f_x(u + iv)| du dv \right)^2 = \left(\int_{R_a} |f_x(p)| dA \right)^2 \leq K a \int_{R_a} \det df dA = Kab.$$

If equality is attained, then $\left(\int_{R_a} |f_x(p)| dA \right)^2 = \int_{R_a} |f_x(p)|^2 dA$, so $|f_x(p)|$, $\|df_p\|$ and $\det df$ are equal to $\frac{b}{a}$ almost everywhere. This is enough to conclude that $f(x + iy) = \frac{b}{a}x + iy$. □

Beltrami and quadratic differentials

In this section, we will describe the tangent and cotangent bundles of the Teichmüller space $\mathcal{T}(S)$ in terms of Beltrami and quadratic differentials, respectively.

Let X be a Riemann surface. We say that a differential form ϕ is a *quadratic differential* on X if it is of the form $\phi(z) dz \otimes dz = \phi(z) dz^2$. That is, if (U, z) and (V, w) are two charts of X, where $U, V \subseteq X$ and $z : U \to \mathbb{C}$, $w : V \to \mathbb{C}$ are holomorphic functions, then for every $p \in U \cap V$:

$$\phi_U(p) = \phi_V(p)(\partial_z w(p))^2.$$

In other words, a quadratic differential is a section of the symmetric square $\mathrm{Sym}^2(\omega)$ of the holomorphic cotangent bundle ω of X (also known as the *canonical bundle*). In this sense, it is a $(2, 0)$-tensor. We say that a quadratic differential is *holomorphic* (resp. *meromorphic*) if its local expression is given by holomorphic (resp. meromorphic) functions.

A dual notion is the following: a differential form μ is a *Beltrami differential* on X if it is of the form $\mu(p) d\bar{z} \otimes (dz)^{-1} = \mu(p) \frac{d\bar{z}}{dz}$. That is, if (U, z) and (V, w) are two charts of X, where $U, V \subseteq X$ and $z : U \to \mathbb{C}$, $w : V \to \mathbb{C}$ are holomorphic functions, then for every $p \in U \cap V$:

$$\mu_U(p) = \mu_V(p) \frac{\partial_{\bar{z}} \overline{w}(p)}{\partial_z w(p)}.$$

In other words, a Beltrami differential is a section of $\bar{\omega} \otimes \omega^*$ (where ω^* should be interpreted as the tensor-inverse of ω, which coincides with the holomorphic tangent bundle). In this sense, it is a $(-1, 1)$-tensor. Of course, since this definition is local, a Beltrami differential does not define a *function* from X to $\mathbb{C}$. Nevertheless, it is easy to see that from the above local expression that $|\mu_U(p)| = |\mu_V(p)|$ for every $p \in X$, so a Beltrami differential defines a function $|\mu| : X \to \mathbb{R}_+$. We will denote by $\|\mu\|_\infty$ the L^∞-norm of $|\mu|$.

The next theorem, that was originally proven by Ahlfors and Bers [AB60], shows the intimate relation between quasiconformal maps and Beltrami differentials.

Theorem 1.1.9 (Measurable Riemann mapping theorem). *Let $U \subseteq \mathbb{C}$ and consider a measurable function $\mu \in L^\infty(U)$ satisfying $\|\mu\|_\infty < 1$. Then, there exists a K-quasiconformal orientation-preserving homeomorphism $f : U \to \mathbb{C}$, with $K = \frac{1+\|\mu\|_\infty}{1-\|\mu\|_\infty}$, solving the Beltrami equation*

$$\frac{\partial f}{\partial \bar{z}} = \mu \frac{\partial f}{\partial z}$$

in the weak sense. Moreover, f is unique modulo composition with holomorphic maps.

An immediate corollary of the previous proposition is the following.

Proposition 1.1.10. *Let X be a Riemann surface and μ be a Beltrami differential on X satisfying $\|\mu\|_\infty < 1$. Given a coordinate chart (U, z), with $U \subseteq X$ and where $z : U \to \mathbb{C}$ is holomorphic, we define $\mu_U : z(U) \to \mathbb{C}$ by $\mu|_U = z^*\left(\mu_U \frac{d\bar{z}}{dz}\right)$. Then, there exists a K-quasiconformal homeomorphism $\zeta_{U,\mu} : z(U) \to \mathbb{C}$, with $K = \frac{1+\|\mu\|_\infty}{1-\|\mu\|_\infty}$, solving the Beltrami equation*

$$\frac{\partial \zeta_{U,\mu}}{\partial \bar{z}} = \mu_U \frac{\partial \zeta_{U,\mu}}{\partial z}$$

in the weak sense. Moreover, the family $(z(U), \zeta_{U,\mu} z)$, where (U, z) ranges over a complex structure of X, is a well-defined complex structure X_μ in the sense that it does not depend on the choice of atlas or on the choice of $\zeta_{U,\mu}$.

The previous proposition suggests a natural question: given $[X, f] \in \mathcal{T}(S)$, what is the subset of $\mathcal{T}(S)$ of the elements $[X_\mu, f]$ obtained by deforming $[X, f]$ via some Beltrami differential μ with $\|\mu\|_\infty < 1$?

To answer this question, we need to state a fundamental theorem of Teichmüller:

Theorem 1.1.11 ([Tei40; Tei43]). *Let (X, f) and (Y, g) be two marked Riemann surfaces on an underlying topological surface S with $\chi(S) < 0$. Then, there exists a quasiconformal orientation-preserving homeomorphism $\sigma : X \to Y$ whose coefficient of quasiconformality is the minimum among all quasiconformal maps in the isotopy class of gf^{-1}. Such a minimizing map is called an extremal map. Conversely, every extremal map $\sigma : X \to Y$ between two marked Riemann surfaces (X, f) and (Y, g) satisfies that $\frac{\|d\sigma_z\|^2}{\det d\sigma_z}$ is constant for all but finitely many $z \in X$. Finally, extremal maps are unique up to composition with conformal maps.*

Now, we can see that if (X, f) and (Y, g) are marked Riemann surfaces, then the extremal map $\sigma : X \to Y$ in the isotopy class of gf^{-1} defines a Beltrami differential μ via the formula $\mu_\sigma = \frac{\overline{(\zeta\sigma)_{\bar{z}}}}{(\zeta\sigma)_z} \frac{d\bar{z}}{dz}$, where (U, z) is a chart on X and $(\sigma(U), \zeta)$ is a chart on Y. This formula for μ is independent of the choice of ζ. Then, we have that $[X_{\mu_\sigma}, f]$ and $[Y, g]$ define the same point in the Teichmüller space. Indeed, the map $\sigma : X_{\mu_\sigma} \to Y$ is biholomorphic and isotopic to gf^{-1}, so σf and g are also isotopic.

The previous paragraph allows us to describe the tangent space $T_{[X,f]}\mathcal{T}(S)$. Indeed, let $\mathcal{B}_X$ be the Banach space of essentially bounded Beltrami differentials on X endowed with the L^∞-norm. Let B_X be the open ball of the $\mu \in \mathcal{B}_X$ such that $\|\mu\|_\infty < 1$. The map $\mu \mapsto [X_\mu, f]$ from

B_X to $\mathcal{T}(S)$ is surjective and, moreover, it is real-analytic [Mor38; AB60]. We have that the derivative $T_0 B_X \to T_{[X,f]}\mathcal{T}(S)$ of this map is surjective and we obtain that $T_{[X,f]}\mathcal{T}(S)$ can be naturally identified with $T_0 B_X$ modulo its kernel. After the usual identification $T_0 B_X = \mathcal{B}_X$ for a Banach space, we obtain a natural identification between $T_{[X,f]}\mathcal{T}(S)$ and $\mathcal{B}_X/\mathcal{B}_X^0$, where $\mathcal{B}_X^0$ is the subspace of *infinitesimally trivial* Beltrami differentials, that is, those induced by derivatives of maps that are isotopic to the identity.

Remark 1.1.12. The spaces $\mathcal{B}_X$ and $\mathcal{B}_X^0$ are infinite-dimensional Banach spaces, but the quotient is finite-dimensional, as expected.

Now, if $\sigma\colon X \to Y$ is an extremal map between (X, f) and (Y, g) then, by definition,

$$d_{\mathcal{T}}([X, f], [X_{s\mu_\sigma}, f]) = \frac{1}{2}\log\frac{1 + |s|\|\mu_\sigma\|_\infty}{1 - |s|\|\mu_\sigma\|_\infty},$$

so, by taking $|s| = \frac{1}{\|\mu_\sigma\|_\infty}\frac{e^{2t}-1}{e^{2t}+1}$, the distance between $[X, f]$ and $[X_{s\mu_\sigma}, f]$ is exactly t. This is what we call a *complex Teichmüller geodesic*. When s is real, we speak of the *Teichmüller geodesic flow*.

To describe the cotangent space to $\mathcal{T}(S)$ at $[X, f]$, we need to find the tensors that pair with the essentially bounded Beltrami differentials $\mathcal{B}_X$. Since these tensors are of type $(-1, 1)$, they pair naturally with tensors of type $(2, 0)$ (as their product is a tensor of type $(1, 1)$, that is, an area form on X). Therefore, we see that the cotangent space $T^*_{[X,f]}\mathcal{T}(S)$ can be naturally identified with the space $\mathbb{Q}_X$ of *integrable* quadratic differentials, that is, quadratic differentials ϕ with a finite L^1-norm $\|\phi\|_1 = \int_X |\phi| dA$. The integrability condition is equivalent to requiring that ϕ is given by meromorphic functions with at most simple poles (located at punctures) in local coordinates. We refer to the cotangent bundle $T^*\mathcal{T}(S)$ as the *Teichmüller space of quadratic differentials* and we denote it by $\mathcal{T}\mathbb{Q}(S)$.

The L^1-norm on the cotangent space induces a weak norm on the tangent space, given by

$$\|\mu\|_* = \sup_{\substack{\phi\in\mathbb{Q}_X \\ \phi \text{ holomorphic}}} \frac{\mathrm{Re}\langle\phi, \mu\rangle}{\|\phi\|_1}.$$

This is the weak norm that induces the Finsler structure of the Teichmüller metric [PS15]: that is, the length of a smooth curve $\theta\colon t \mapsto [X_t, f_t]$, with $t \in [0, 1]$, on the Teichmüller space can be computed $\ell_{\mathcal{T}}(\theta) = \int_0^1 \|\theta'(t)\|_* dt$. It can be checked that the Teichmüller geodesic flow has unit speed.

1.1.4 The moduli space of Riemann surfaces

While the Teichmüller space parametrises the marked Riemann surfaces, the moduli space parametrises Riemann surfaces themselves. Therefore, it can be defined by "forgetting the marking". To make this precise, we need to introduce the mapping class group of a surface.

Let S be a (possibly punctured) topological surface. Then, its *mapping class group* $\mathrm{Mod}(S)$ is its set of orientation-preserving homeomorphisms modulo isotopy. If S has the structure of a smooth manifold (a complex structure, for instance) we may also define this group as its set of orientation-preserving *diffeomorphisms* modulo smooth isotopy. The fact that these two groups are equal follows from a theorem of Munkres [Mun60], Smale and Whitehead [Whi61]:

Theorem 1.1.13. *Let S be a (possibly punctured) smooth closed surface. Then, every homeomorphism of S is isotopic to a diffeomorphism of S.*

We have that two self-maps ξ, η of S are isotopic if and only if $\eta\xi^{-1}$ is isotopic to the identity. Thus, $\mathrm{Mod}(S)$ can also be defined as $\mathrm{Homeo}^+(S)/\mathrm{Homeo}_0^+(S)$ or $\mathrm{Diffeo}^+(S)/\mathrm{Diffeo}_0^+(S)$, where $\mathrm{Homeo}_0^+(S)$ (resp. $\mathrm{Diffeo}_0^+(S)$) is the set of orientation-preserving homeomorphisms (resp. diffeomorphisms) which are isotopic (resp. smoothly isotopic) to the identity.

The mapping class group acts on the Teichmüller space $\mathcal{T}(S)$ by isometries for the Teichmüller metric. Indeed, given $[X, f], [Y, g] \in \mathcal{T}(S)$ we define $\xi \cdot [X, f] = [X, f\xi^{-1}]$. It is easy to see that if η is isotopic to ξ, then $[X, f\xi^{-1}] = [X, f\eta^{-1}]$, so this action is well-defined. Moreover, $gf^{-1} = (g\xi^{-1})(f\xi^{-1})^{-1}$, so $d_{\mathcal{T}}([X, f], [Y, g]) = d_{\mathcal{T}}([X, f\xi^{-1}], [Y, g\xi^{-1}])$. We can also define natural actions on $T\mathcal{T}(S)$ and $T^*\mathcal{T}(S)$. Indeed, if $[X, f] \in \mathcal{T}(S)$ and $\mu \in \mathcal{B}_X$ is a Beltrami differential on X, then we define $\xi \cdot ([X, f], \mu) = ([X, f\xi^{-1}], \mu)$. Analogously, given an integrable quadratic differential $\phi \in \mathbb{Q}_X$ on X, then we define $\xi \cdot ([X, f], \phi) = ([X, f\xi^{-1}], \phi)$.

We can now define the moduli space of Riemann surfaces by $\mathcal{M}(S) = \mathcal{T}(S)/\mathrm{Mod}(S)$. The covering map $\mathcal{T}(S) \to \mathcal{M}(S)$ has infinite degree. Moreover, the action of $\mathrm{Mod}(S)$ on $\mathcal{T}(S)$ is properly discontinuous, but it fails to be free at some specific points. Such points are said to possess nontrivial *automorphisms*. Nevertheless, the moduli space has a natural *orbifold* structure inherited from the manifold structure of the Teichmüller space. Furthermore, it is possible to find a finite-index subgroup of $\mathrm{Mod}(S)$ that acts freely on $\mathcal{T}(S)$, so $\mathcal{M}(S)$ is finitely covered by a smooth manifold. Our discussion on the tangent and cotangent spaces of $\mathcal{T}(S)$ carries over to this finite cover of $\mathcal{M}(S)$ and, thus, it is "almost" valid in $\mathcal{M}(S)$. We will usually ignore these technical details in this exposition. We refer to the cotangent bundle of $T^*\mathcal{M}(S)$ as the *moduli space of quadratic differentials* and we denote it by $\mathbb{Q}(S)$.

The Teichmüller metric $d_{\mathcal{T}}$ descends to a well-defined metric $d_{\mathcal{M}}$ on $\mathcal{M}(S)$, which we will also call the *Teichmüller metric* by a slight abuse of notation. We define the *Teichmüller complex geodesics* and the *Teichmüller geodesic flow* on $\mathcal{M}(S)$ as the corresponding projections of these objects from $\mathcal{T}(S)$ to $\mathcal{M}(S)$.

The study of the Teichmüller geodesic flow and, more generally, of an $\mathrm{SL}(2, \mathbb{R})$-action on the cotangent bundle of the moduli space of Riemann surfaces is the main topic of Teichmüller dynamics, which is introduced in the next section.

1.2 Teichmüller dynamics

The main actor in the domain of Teichmüller dynamics is an action of the group $SL(2, \mathbb{R})$ on the cotangent bundle of the moduli space of Riemann surfaces, that is, on the moduli space of quadratic differentials. The Teichmüller geodesic flow, that we have already defined, can be seen as the action of the diagonal subgroup of this bigger group. While the Teichmüller geodesic flow may seem more natural at first in view of Teichmüller's theorem, the $SL(2, \mathbb{R})$-action has very interesting dynamical, algebraic and geometric properties.

We start this section by defining translation structures and relating them to quadratic differentials on Riemann surfaces. The definition of such structures follows cleanly from another theorem by Teichmüller, which is also formulated at the beginning of the first part. It is a very natural setting in which to define the Teichmüller flow and, more generally, the $SL(2, \mathbb{R})$-action.

In the second part of this chapter, we will give an equivalent definition of a translation structure or a quadratic differential in more combinatorial terms by using polygon representations. We will discuss how the moduli space of quadratic differentials is naturally stratified by the order of the singularities—zeroes or poles—of the quadratic differential. Such strata are not connected, but they always possess at most three connected components that we will discuss in some detail. The combinatorial definition is very useful to produce concrete examples of quadratic differentials belonging to a given connected component, as we will show when we present permutation representatives. We will finish this part by presenting a simple way to parametrise a component of a stratum of the Teichmüller space of quadratic differentials, by using the so-called period coordinates. These coordinates provide a coherent way to connect homology groups of nearby quadratic differentials, via the Gauss–Manin connection, and to define natural "Lebesgue" measures on these spaces, which are called the Masur–Veech measures.

In the third part of this chapter, we will focus on the actual dynamics on the moduli space of quadratic differentials. One of the most fundamental results, due to Masur and Veech, is the ergodicity of the Teichmüller flow with respect to the Masur–Veech measures. Afterwards, we will define the Kontsevich–Zorich cocycle—the main subject of study of this thesis—which encodes the homological part of a given action on the moduli space of quadratic differentials. We will discuss the groups generated by such homological actions, which are the monodromy groups or the Rauzy–Veech groups depending on the action we consider.

In the fourth and last part of this chapter, we develop some of the theory of square-tiled surfaces, which are very specific examples of quadratic differentials that can be studied in almost purely combinatorial terms. In some sense, they can be regarded as the "integer points" of some strata of the moduli space of quadratic differentials, and have, thus, zero measure. Nevertheless, they exhibit a very rich behaviour and are a plentiful source of examples.

1.2.1 Flat structures

We will start by presenting another theorem of Teichmüller which allows us to interpret quadratic differentials in a combinatorial way.

Theorem 1.2.1 ([Tei40; Tei43]). *Let (X, f) and (Y, g) be two marked Riemann surfaces and let $\sigma : X \to Y$ be an extremal map in the isotopy class of gf^{-1}. Then, there exist atlases on X and Y which are compatible with their complex structures such that:*

1. *the transition maps are of the form $\pm z + c$ up to neighbourhoods of finitely many points;*
2. *the major (resp. minor) axes of the infinitesimal ellipses which are the images of infinitesimal circles by σ are aligned with the horizontal (resp. vertical) foliation;*
3. *in terms of these coordinates, the horizontal direction is expanded by an uniform factor of $\sqrt{K}$ and the vertical direction is contracted by an uniform factor of $\sqrt{1/K}$, where K is the coefficient of quasiconformality of σ.*

Since the Teichmüller geodesic flow can be understood in terms of extremal maps, we obtain that, in some sense, it acts by expanding the horizontal direction and by contracting the vertical direction. To make this idea precise, we need to introduce flat structures and flat surfaces.

Let X be a closed surface. We say that a *flat structure* on X is an atlas transition maps of the form $\pm\zeta + c$ up to neighbourhoods of a finite set Σ. Observe that a flat structure is, in particular, a Riemann surface structure on the punctured surface $X \setminus \Sigma$ because maps of the form $\pm\zeta + c$ are holomorphic. A surface endowed with a flat structure is called a *flat surface*.

A flat structure induces a flat Riemannian metric on $X \setminus \Sigma$, that is, a Riemannian metric whose curvature is 0. Such a metric defines a geodesic flow, which is usually called the *linear flow*, *translation flow* or *straight line flow* on the surface (since, as the curvature vanishes, geodesics are straight lines). Moreover, the flat structure defines a quadratic differential as the pullback of $d\zeta^2$ defined on $\mathbb{C}$ by the coordinate charts. This leads to a well-defined quadratic differential ϕ on $X \setminus \Sigma$ because of the assumption on the transition maps. Indeed, if (U, z) is a coordinate chart of the flat structure, then $\phi = dz^2$ on U. If (V, w) is another overlapping coordinate chart, then

$$\phi = (\partial_z w)^2 dw^2 = (\partial_\zeta(\pm\zeta + c))^2 dw^2 = dw^2,$$

in $U \cap V$. Now, we can extend the definition to neighbourhoods of Σ. Given $p \in \Sigma$, we have two possible cases. First, if it is possible to extend the definition to p by continuity, we have that $\phi = z^k dz^2$, with $k \geq 0$, for some local coordinate z at p. If $k > 0$, then p is said to be a *zero* of ϕ. Otherwise, if ϕ cannot be extended to p and if we assume ϕ to be integrable, we have that $\phi = z^{-1} dz^2$ for some local coordinate z at p. We say that p is a *pole*. To speak collectively about zeros and poles, we use the term *singularity*. We obtain, thus, that ϕ is a meromorphic quadratic differential with at most simple poles, that is, $\phi \in \mathbb{Q}_X$.

Conversely, assume that X is a closed Riemann surface and that $\phi \in \mathbb{Q}_X$ is an integrable quadratic differential on X. Then, ϕ defines a flat structure simply by taking an atlas for which

ϕ is the pullback of dz^2 outside of neighbourhoods of finitely many points Σ. Since in the intersection of charts (U, z) and (V, w) we have that $\phi = dz^2 = (\partial_z w)^2 dw^2$, we obtain that the transition maps must be of the form $\pm\zeta + c$. Near $p \in \Sigma$, it can be shown that $\phi = z^k dz$ with $k \geq -1$. If $k > 0$ then $\phi(p) = 0$ and if $k = -1$ then p is a simple pole for ϕ. As before, we collectively call the zeros and poles of ϕ its *singularities*. Observe that the resulting transition maps are biholomophic away from a singularity, so we obtain a Riemann surface structure on the punctured surface $X \setminus \Sigma$.

Remark 1.2.2. When we speak about a flat surface M, we usually refer to a surface without punctures. Nevertheless, M is in general *not* a Riemann surface because the the local coordinates are ramified at the singularities. However, there exists a finite set Σ such that $M \setminus \Sigma$ is a punctured Riemann surface.

The flat Riemannian metric on $X \setminus \Sigma$ cannot be extended to X in the presence of singularities. Nevertheless, any straight line geodesic ray can be extended either to an infinitely long geodesic ray or to a geodesic ray ending at a singularity. A geodesic joining two singularities is said to be a *saddle connection*.

The $\mathrm{SL}(2, \mathbb{R})$-action on the moduli space of quadratic differentials

Flat structures allow an $\mathrm{SL}(2, \mathbb{R})$-action which is defined in the following way. Let $g \in \mathrm{SL}(2, \mathbb{R})$ and M be a flat surface. Then, we define $g \cdot M$ to be the flat surface whose flat structure is given by the charts $(U, g \cdot z)$, where g acts on $\mathbb{C}$ via the identification $\mathbb{C} \simeq \mathbb{R}^2$ and (U, z) is a chart of the flat structure of M. This definition indeed produces a flat structure: if $(V, g \cdot w)$ is another overlapping chart, then the transition map between U and V is:

$$g \cdot (wz^{-1})(g^{-1} \cdot \zeta) = g \cdot (\pm g^{-1} \cdot \zeta + c) = \pm\zeta + gc,$$

where $wz^{-1}(\zeta) = \pm\zeta + c$. Therefore, transition maps of $g \cdot M$ are also of the required form.

The Teichmüller geodesic flow can be now interpreted as a subgroup of this $\mathrm{SL}(2, \mathbb{R})$-action. Indeed, given $[X, f] \in \mathcal{T}(S)$ and $\mu \in B_X$, the Teichmüller geodesic flow produces another point $[X_\mu, f] \in \mathcal{T}(S)$. If $\sigma \colon X \to X_\mu$ is an extremal map in the isotopy class of $ff^{-1} = \mathrm{Id}$, then by Theorem 1.2.1, there exist flat structures on X and X_μ such that σ is given by the matrix $\mathrm{diag}(\sqrt{K}, \sqrt{1/K})$ in terms of the $\mathrm{SL}(2, \mathbb{R})$-action for some $K > 1$. If the Teichmüller distance between $[X, f]$ and $[X_\mu, f]$ is t, then $K = e^{2t}$. Thus, we define the *Teichmüller geodesic flow* on the Teichmüller space of quadratic differentials as the flow given by the diagonal subgroup of $\mathrm{SL}(2, \mathbb{R})$ of matrices of the form $\mathrm{diag}(e^t, e^{-t})$, where $t \in \mathbb{R}$. Analogously, the *Teichmüller geodesic flow* on the moduli space of quadratic differentials is the projection of this flow from $\mathcal{T}\mathbb{Q}(S)$ to $\mathbb{Q}(S)$.

Abelian and quadratic differentials

A quadratic differential ϕ can be either orientable or not. The first case is equivalent to saying that $\phi = \omega^2$, where ω is a *holomorphic* nonzero 1-form on X. We call such a 1-form ω an *Abelian differential* on X. We identify a holomorphic 1-form on a Riemann surface with its global square, and thus define the *Teichmüller space of Abelian differentials* $\mathcal{T}\mathcal{H}(S)$ as such subbundle of $T^*\mathcal{T}(S)$. Analogously, we also define the *moduli space of Abelian differentials* $\mathcal{H}(S)$.

Moreover, given any nonorientable quadratic differential ϕ it is possible to build a canonical (possibly ramified) double cover of it which is an Abelian differential [Mas82]. Thus, any nonorientable quadratic differential can be thought of as the quotient of an Abelian differential by an appropriate involution. For the following explicit construction, we follow a discussion by Lanneau [Lan04, Construction 1]. Let M be a flat surface and let ϕ be the corresponding quadratic differential and Σ be its (finite) set of singularities. We assume ϕ to be nonorientable. For each chart (U, z) of $M \setminus \Sigma$, we consider two disjoint copies $U^\pm$. If the charts (U, z) and (V, w) overlap, then $\phi_U(p) = \phi_V(p)(\partial_z w(p))^2$. We define two functions $\phi_U^\pm$, one for each branch of $\sqrt{\phi_U}$. Now, we identify the part $U^\pm$ corresponding to $U \cap V$ with the part of V^+ corresponding to $U \cap V$ in such a way that

$$\phi_U^\pm(p) = \phi_V^+(p)\partial_z w(p).$$

The part V^- is identified analogously. We obtain a punctured Riemann surface $\widetilde{M} \setminus \widetilde{\Sigma}$ endowed with an Abelian differential ω. It can be shown that the punctures of $\widetilde{M} \setminus \widetilde{\Sigma}$ can be filled, producing a closed Riemann surface $\widetilde{M}$ together with an involution $\iota \colon \widetilde{M} \to \widetilde{M}$ given by exchanging U^+ and U^-. The quotient $\widetilde{M}/\iota$ can be identified with M in a natural way: the pullback of ϕ by the quotient map is exactly ω^2.

1.2.2 Combinatorics and topology of the moduli space of flat surfaces

We will now formulate the definition of flat surface in a more combinatorial way. We say that a *flat surface* M is a finite collection of polygons with an even number of sides which are identified in pairs via translations and rotations by π. Only "opposite" sides are identified, that is, the resulting topological surface is orientable. A flat surface induces a flat structure. If the identifications are only by translations (and no rotations by π), the flat structure induces an Abelian differential and such a flat surface is usually called a *translation surface*. Otherwise, the flat structure induces a nonorientable quadratic differential and the surface is said to be a *half-translation surface*. Collectively, these surfaces are called *flat surfaces*. Observe that if two sides are identified, then they are necessarily parallel and have the same length. Conversely, a flat structure defines such a collection of polygons. This can be proved by triangulating the surface using saddle connections, cutting along such segments and then gluing the resulting triangles.

An element of the *Teichmüller space of flat surfaces* is an equivalence class of such collections of polygons for cut-and-paste operations, with a marking to a fixed topological surface. That is, two collections are equivalent whenever it is possible to cut pieces of the first along straight

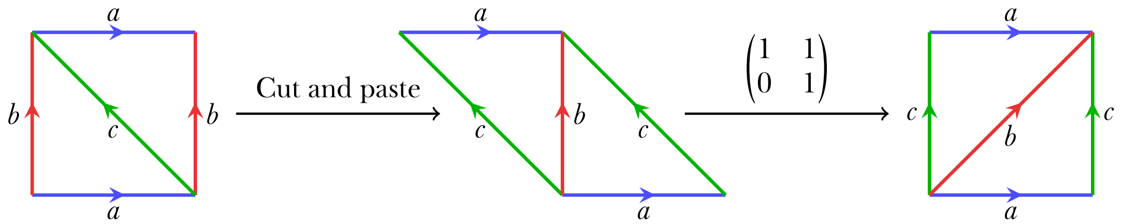


Figure 1.1: An example of a polygon representation of a translation surface (left). When a cut-and-paste operation is applied, we obtain the exact same element of the Teichmüller space of translation surfaces (middle). Then, if the $\mathrm{SL}(2,\mathbb{R})$-action is applied we obtain a different element on the Teichmüller space of translation surfaces as the composition of the markings is a Dehn twist, which is not isotopic to the identity (right). Nevertheless, these three polygon representations correspond to the same element of the moduli space of translation surfaces.

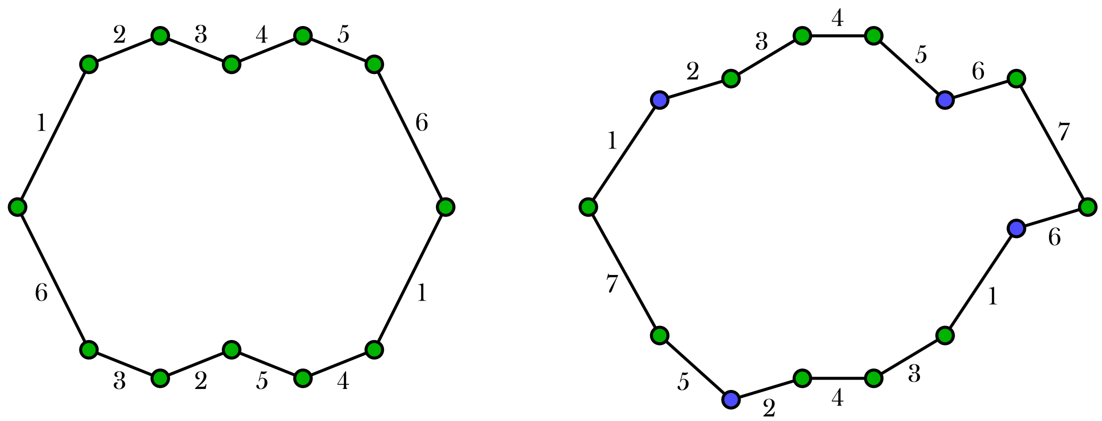


Figure 1.2: Examples of polygon representations of translation surfaces and their singularities. The numbers on the edges label the sides that are identified. The surface on the left has a single singularity of angle 10π (in green), while the surface on the right has two singularities of angles 8π (in green) and 4π (in blue).

lines and reglue them to produce the latter in a way that is compatible with the marking. An element of the *moduli space of flat surfaces* is an equivalence class of collections of polygons, with no marking whatsoever. The Teichmüller (resp. moduli) spaces of translation surfaces and of half-translation surfaces are defined as the subsets of the Teichmüller (resp. moduli) space of flat surfaces consisting of Abelian and nonorientable quadratic differentials, respectively. See Figure 1.1.

We obtain, thus, three equivalent definitions of a flat surface: a surface endowed with a flat structure, a punctured Riemann surface endowed with an Abelian or quadratic differential and a collection of polygons with some side identifications. The third definition is the way most people think about flat surfaces since it easily allows to exhibit explicit examples. See Figure 1.2 and Figure 1.3.

Let M be a flat surface and let $p \in M$ be a singularity. If M is a translation surface, then

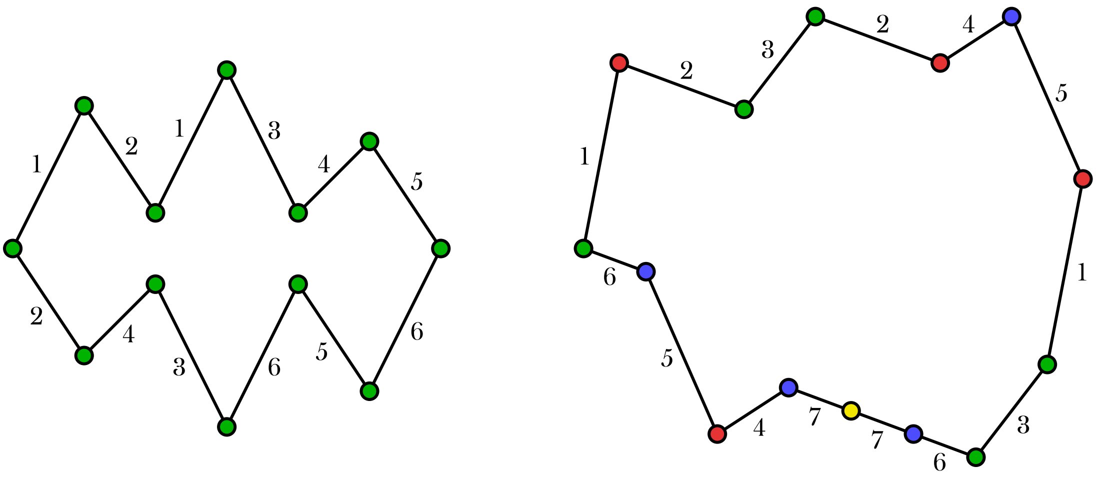


Figure 1.3: Examples of polygon representations of half-translation surfaces and their singularities. The numbers on the edges label the sides that are identified. The surface on the left has a single singularity of angle 10π (in green), while the surface on the right has four singularities of angles 4π (in green), 4π (in blue), 3π (in red) and π (in yellow).

the total angle around p is $2(k+1)\pi$, where $k \geq 0$ is an integer. If M is a half-translation surface, then the total angle around p is $(k+2)\pi$, where $k \geq -1$ is an integer. In both cases, k is called the *order* of p and denoted $\operatorname{ord}(p)$. For a *regular point*, that is, a point with total angle 2π, it is consistent to define its order to be 0. The Riemann–Hurwitz theorem then implies that $\sum_{p \in M} \operatorname{ord}(p) = 2g - 2$ in the case of translation surfaces, and that $\sum_{p \in M} \operatorname{ord}(p) = 4g - 4$ in the case of half-translation surfaces, where g is the genus of the underlying topological surface. Observe that these sums are actually finite sums as the order vanishes outside of the finite set of singularities. See Figure 1.2 and Figure 1.3 for some examples.

Let M be a half-translation surface and let ϕ be its corresponding quadratic differential. Let $\widetilde{M}$ be its orientable double cover. Near a zero of even degree $2k$ of ϕ, the differential can be written as $z^{2k} dz^2$. Therefore, we get two distinct branches $\pm z^k dz$ in $\widetilde{M}$. This means that a zero of order $2k$ in M corresponds to two zeroes of order k in $\widetilde{M}$. On the other hand, every singularity p of odd order $2k-1$ of ϕ is a branch point for the covering. Thus, the angle at the corresponding singularity $\tilde{p}$ of $\widetilde{M}$ is twice the angle at p. Since the angle at p is $(2k+1)\pi$, the angle at $\tilde{p}$ is $2(2k+1)\pi$ and we obtain that $\operatorname{ord}(\tilde{p}) = \operatorname{ord}(p) + 1$.

Permutations and generalised permutations

A widely-used combinatorial way to produce polygons representing flat surfaces is to use (generalised) permutation representatives. We will start with the more classical case of permutations and translation surfaces, and then we will discuss the case of generalised permutations and half-translation surfaces.

Let $\mathcal{A}$ be a finite alphabet with $d \geq 2$ letters. A *permutation* is a pair of bijections $\pi = (\pi_t, \pi_b)$

from $\mathcal{A}$ to $\{1, \ldots, d\}$. Here, the labels t and b stand for "top" and "bottom", respectively. We interpret these bijections as an order on $\mathcal{A}$: the first letter on top is $\pi_{\mathrm{t}}^{-1}(1)$, the second letter is $\pi_{\mathrm{t}}^{-1}(2)$ and so on. Analogously, the first letter on bottom is $\pi_{\mathrm{b}}^{-1}(1)$, the second is $\pi_{\mathrm{b}}^{-1}(2)$ and so on. We set $\alpha_{\varepsilon,k} = \pi_{\varepsilon}^{-1}(k)$, where $\varepsilon \in \{\mathrm{t}, \mathrm{b}\}$, and we usually write a permutation as a table

$$\pi = \begin{pmatrix} \alpha_{\mathrm{t},1} & \alpha_{\mathrm{t},2} & \cdots & \alpha_{\mathrm{t},d} \\ \alpha_{\mathrm{b},1} & \alpha_{\mathrm{b},2} & \cdots & \alpha_{\mathrm{b},d} \end{pmatrix}.$$

Remark 1.2.3. While a permutation according to this definition defines a unique permutation in the usual sense of abstract algebra, the converse is not true. Indeed, a permutation in the sense of the theory of flat surfaces carries a distinguished order in which the elements of the alphabet are written.

We say that a permutation $\pi = (\pi_{\mathrm{t}}, \pi_{\mathrm{b}})$ is *irreducible* if $\pi_{\mathrm{t}}^{-1}(\{1, \ldots, j\}) \neq \pi_{\mathrm{b}}^{-1}(\{1, \ldots, j\})$ for any $1 \leq j < d$. Moreover, we say that π is *degenerate* if there exists $1 \leq j < d$ such that one of the following three conditions holds:

$$\pi_{\mathrm{b}}(\alpha_{\mathrm{t},j}) = d, \quad \pi_{\mathrm{b}}(\alpha_{\mathrm{t},j+1}) = 1 \quad \text{and} \quad \pi_{\mathrm{b}}(\alpha_{\mathrm{t},1}) = \pi_{\mathrm{b}}(\alpha_{\mathrm{t},d}) + 1;$$

$$\pi_{\mathrm{b}}(\alpha_{\mathrm{t},j+1}) = 1 \quad \text{and} \quad \pi_{\mathrm{b}}(\alpha_{\mathrm{t},1}) = \pi_{\mathrm{b}}(\alpha_{\mathrm{t},j}) + 1;$$

or

$$\pi_{\mathrm{b}}(\alpha_{\mathrm{t},j}) = d \quad \text{and} \quad \pi_{\mathrm{b}}(\alpha_{\mathrm{t},j+1}) = \pi_{\mathrm{b}}(\alpha_{\mathrm{t},d}) + 1.$$

Otherwise, π is said to be *nondegenerate*. From now on, we will always assume that all permutations are both irreducible and nondegenerate.

A *suspension datum* for a permutation π is a collection $\{\zeta_\alpha\}_{\alpha\in\mathcal{A}}$ of complex numbers satisfying:

- $\mathrm{Re}(\zeta_\alpha) > 0$ for each $\alpha \in \mathcal{A}$;
- $\sum_{1\leq j\leq i} \mathrm{Im}(\zeta_{\alpha_{\mathrm{t},j}}) > 0$ for each $1 \leq i < d$;
- $\sum_{1\leq j\leq i} \mathrm{Im}(\zeta_{\alpha_{\mathrm{b},j}}) < 0$ for each $1 \leq i < d$;
- $\sum_{1\leq i\leq d} \zeta_{\alpha_{\mathrm{t},i}} = \sum_{1\leq i\leq d} \zeta_{\alpha_{\mathrm{b},i}}$.

A permutation together with a suspension datum defines a polygon in $\mathbb{C}$ whose "top" sides are the $\{\zeta_\alpha\}_{\alpha\in\mathcal{A}}$ in the order given by π_{t} and whose "bottom" sides are the $\{\zeta_\alpha\}_{\alpha\in\mathcal{A}}$ in the order given by π_{b}. Nevertheless, this polygon may intersect itself. We say that the suspension datum is *suitable* or that it defines a *suitable polygon* if the associated polygon does not intersect itself. In this case, we may label the sides ζ_α with the label α, identify the equally-labelled sides and obtain a valid translation surface.

Remark 1.2.4. It is sometimes possible to define valid translation surfaces even for nonsuitable suspension data by using Veech's zippered rectangles construction [Vee82]. However, we will not describe this construction in this thesis.

There exists a suspension datum defining a suitable polygon if and only if the permutation is irreducible. Moreover, there exists a "canonical" way to produce such a polygon: the suspension datum given by $\zeta_\alpha = 1 + i(\pi_{\mathrm{b}}(\alpha) - \pi_{\mathrm{t}}(\alpha))$ for each $\alpha \in \mathcal{A}$. This suspension datum further satisfies that $\sum_{1 \leq i \leq \ell} \zeta_{\alpha_{\mathrm{t},i}} = \sum_{1 \leq i \leq d} \zeta_{\alpha_{\mathrm{b},i}} = d$. The resulting polygon is called *Masur's polygon* and we will denote it by P_π. See Figure 1.2 for some examples of this polygon construction for the permutations $\left(\begin{smallmatrix} 1 & 2 & 3 & 4 & 5 & 6 \\ 6 & 3 & 2 & 5 & 4 & 1 \end{smallmatrix}\right)$ and $\left(\begin{smallmatrix} 1 & 2 & 3 & 4 & 5 & 6 & 7 \\ 7 & 5 & 2 & 4 & 3 & 1 & 6 \end{smallmatrix}\right)$ (the vertical direction was scaled for aesthetic reasons). After identifying equally-labelled sides, we obtain a translation surface that we denote by M_π. Moreover, we denote by Σ_π the points of M_π obtained by projecting the vertices of P_π to M_π. This set contains the set of singularities, but it may also contain some regular points. Therefore, we call it the set of *marked points*.

We will now generalise classical permutations to the more complicated case of half-translation surfaces. This formalism was originally defined by Boissy and Lanneau [BL09]. Let $\mathcal{A}$ be a finite set of cardinality $d \geq 2$ and ℓ, m be positive integers satisfying $\ell + m = d$. A *generalised permutation* of type (ℓ, m) is a two-to-one map $\pi\colon \{1, \ldots, 2d\} \to \mathcal{A}$. We usually write such a map by a table

$$\pi = \begin{pmatrix} \pi(1) & \pi(2) & \cdots & \pi(\ell) \\ \pi(\ell+1) & \pi(\ell+2) & \cdots & \pi(\ell+m) \end{pmatrix}.$$

An involution $\sigma\colon \{1, \ldots, 2d\} \to \{1, \ldots, 2d\}$ is defined naturally from a generalised permutation by the rules $\sigma(i) \neq i$ and $\pi(\sigma(i)) = \pi(i)$ for every $i \in \{1, \ldots, 2d\}$. That is, $\{i, \sigma(i)\}$ are the two positions of the letter $\pi(i) = \pi(\sigma(i))$. We say that a letter is *duplicate* if both of its occurrences are in the same row. We denote by $\mathcal{A} \uplus \mathcal{A}$ the multiset $\{\pi(1), \pi(2), \ldots, \pi(2d)\}$ of cardinality $2d$.

We can treat "genuine" permutations as special cases of generalised permutations. Indeed, a *permutation* may be regarded as a generalised permutation such that $\ell = m$ and $\sigma(i) > \ell$ for every $1 \leq i \leq \ell$, that is, having no duplicate letters. We say that a generalised permutation is a strict generalised permutation if it is not a permutation. We will also assume the following [BL09, Convention 2.7]:

Convention 1. *Every strict generalised permutation contains duplicate letters in both rows.*

The importance of this convention lies in that it is necessary for the existence of a suspension of a strict generalised permutation.

A *decomposition* of a generalised permutation π is a way of writing it as

$$\pi = \left(\begin{array}{c|c|c} F_{\mathrm{tl}} & *\,*\,* & F_{\mathrm{tr}} \\ \hline F_{\mathrm{bl}} & *\,*\,* & F_{\mathrm{br}} \end{array}\right)$$

where $F_{\mathrm{tl}}, F_{\mathrm{tr}}, F_{\mathrm{bl}}, F_{\mathrm{br}}$ are (possibly empty) subsets of $\mathcal{A}$ or $\mathcal{A} \uplus \mathcal{A}$. This notation means that there exist $1 \leq i_1 \leq i_2 \leq \ell < i_3 \leq i_4 \leq \ell + m$ such that

- $F_{\mathrm{tl}} = \{\pi(1), \ldots, \pi(i_1)\}$;
- $F_{\mathrm{tr}} = \{\pi(i_2), \ldots, \pi(\ell)\}$;
- $F_{\mathrm{bl}} = \{\pi(\ell+1), \ldots, \pi(i_3)\}$;

- $F_{\mathrm{br}} = \{\pi(i_4), \ldots, \pi(\ell+m)\}$.

Once a decomposition is clear from context, we refer to F_{tl}, F_{tr}, F_{bl}, F_{br} as the top-left, top-right, bottom-left and bottom-right corners of π, respectively.

Let π be a strict generalised permutation. We say that π is *reducible* if there exists a decomposition

$$\pi = \left(\begin{array}{c|c|c} A \cup B & *\,*\,* & D \cup B \\ \hline A \cup C & *\,*\,* & D \cup C \end{array}\right)$$

where A, B, C, D are disjoint (possibly empty) subsets of $\mathcal{A}$ satisfying one of the following conditions:

- no corner is empty;
- there is exactly one empty corner and it is on the left;
- there are exactly two empty corners and they are on the same side.

Otherwise, we say that it is *irreducible*.

Remark 1.2.5. The definition of irreducibility for generalised permutation is not well-adapted for (genuine) permutations. Thus, in that case we will use the classical definition of irreducibility.

One has that a generalised permutation stems from the directional flow of a quadratic differential on a Riemann surface or, equivalently, admits a suspension datum (defined below) if and only if it is irreducible and satisfies Convention 1 [BL09, Theorem 3.2]. From now on, we will always assume that a generalised permutation satisfies these properties unless explicitly stated otherwise.

A *suspension datum* for a generalised permutation π is a collection $\{\zeta_\alpha\}_{\alpha\in\mathcal{A}}$ of complex numbers satisfying:

- $\mathrm{Re}(\zeta_\alpha) > 0$ for each $\alpha \in \mathcal{A}$;
- $\sum_{1\leq j\leq i} \mathrm{Im}(\zeta_{\pi(j)}) > 0$ for each $1 \leq i < \ell$;
- $\sum_{1\leq j\leq i} \mathrm{Im}(\zeta_{\pi(\ell+j)}) < 0$ for each $1 \leq i < m$;
- $\sum_{1\leq i\leq \ell} \zeta_{\pi(i)} = \sum_{1\leq i\leq m} \zeta_{\pi(\ell+i)}$.

As in the case of permutations, a suspension datum may not necessarily define a suitable polygon. That is, the broken lines defined by the suspension datum may intersect at points different from 0 and $\sum_{1\leq i\leq \ell} \zeta_{\pi(i)} = \sum_{1\leq i\leq m} \zeta_{\pi(\ell+i)}$. Nevertheless, it is always possible to construct another suspension datum from ζ which defines a suitable polygon [BL09, Lemma 2.12]. For a generalised permutation π, we choose any suspension datum ζ that admits a suitable polygon and define P_π to be such polygon and M_π to be the half-translation surface obtained by identifying the equally-labelled sides of P_π by translations and/or central symmetries and we define Σ_π to be the points in M_π that lie at the vertices. We put $e_\pi = \sum_{1\leq i\leq \ell} \zeta_{\pi(i)} = \sum_{1\leq i\leq m} \zeta_{\pi(\ell+i)}$. See Figure 1.3 for some examples of possible polygons for the generalised permutations $\left(\begin{smallmatrix} 1 & 2 & 1 & 3 & 4 & 5 \\ 2 & 4 & 3 & 6 & 5 & 6 \end{smallmatrix}\right)$ and $\left(\begin{smallmatrix} & 1 & 2 & 3 & 2 & 4 & 5 & \\ 6 & 5 & 4 & 7 & 7 & 6 & 3 & 1 \end{smallmatrix}\right)$. The arbitrary choice of P_π is not a problem, since the notions that we will define and use are homological. Moreover, we denote by Σ_π the points of M_π obtained by projecting the vertices of P_π to M_π and call it the set of *marked points*.

Stratification of the moduli space of flat surfaces

The moduli space of flat surfaces is naturally stratified according to the cardinalities and orders of the singularities. Indeed, the $\mathrm{SL}(2,\mathbb{R})$-action clearly preserves such combinatorial data because an angle which is an integer multiple of π does not change under linear maps. We denote such strata by $\mathcal{T}\mathcal{H}(\kappa) \subseteq \mathcal{T}\mathcal{H}(S)$ and $\mathcal{H}(\kappa) \subseteq \mathcal{H}(S)$ (resp. $\mathcal{T}\mathbb{Q}(\kappa) \subseteq \mathcal{T}\mathbb{Q}(S)$ and $\mathbb{Q}(\kappa) \subseteq \mathbb{Q}\mathcal{H}(S)$) for translation surfaces (resp. half-translation surfaces), where κ is a list of integers corresponding to the orders of the singularities and adding up to $2g-2$ (resp. $4g-4$). When using this notation, we distinguish between orientable and nonorientable quadratic differentials. That is, the notations $\mathcal{T}\mathbb{Q}(\kappa)$ and $\mathbb{Q}(\kappa)$ are reserved exclusively for quadratic differentials that are not squares of Abelian differentials. Therefore, strata are disjoint.

Using this notation and following our previous discussion about the orders of the singularities, every stratum $\mathbb{Q}(2m_1-1,\cdots,2m_s-1,2m_{s+1},\cdots,2m_n)$ of quadratic differentials is naturally covered by Abelian differentials belonging to $\mathcal{H}(2m_1,\cdots,2m_s,m_{s+1},m_{s+1},\cdots,m_n,m_n)$.

The strata are not necessarily connected (and some of them are even empty!). Indeed, each stratum may be either empty or have one, two or three distinct connected components. The components of Abelian strata were classified by Kontsevich and Zorich [KZ03], while the connected components of quadratic strata were classified by Lanneau [Lan08] and Chen and Möller [CM14]. These classifications rely on several invariants that we will discuss now.

The first invariant is hyperellipticity. We say that a flat surface M is *hyperelliptic* if there exists an involution $\iota\colon M \to M$ with exactly $2g+2$ fixed points. This is equivalent to saying that M/ι is a topological sphere. Hyperellipticity is an "extra symmetry" on a surface that may be broken when M is deformed. Nevertheless, this extra symmetry is sometimes forced by the combinatorial constraints of the component of the stratum and, therefore, it holds for every flat surface in such component. In such a case, we speak about a *hyperelliptic component* of a stratum. It is an important result that every genus-2 translation surface is hyperelliptic.

The second invariant is spin parity. It is only defined for strata whose singularities have only even orders, that is, strata of the form $\mathcal{H}(2m_1,\ldots,2m_n)$. Let M be a translation surface in such a stratum and let $Q\colon H_1(M\setminus\Sigma;\mathbb{Z}/2\mathbb{Z}) \to \mathbb{Z}/2\mathbb{Z}$ the quadratic form defined as follows. For any $u \in H_1(M\setminus\Sigma;\mathbb{Z}/2\mathbb{Z})$, we take a smooth simple closed curve θ whose modulo-two homology class is u. Then, we define $Q(u) = \operatorname{ind}(\theta)+1$, where $\operatorname{ind}(\theta)$ is the degree of the Gauss map (also known as index or turning number) of θ. This quantity is well-defined because of the assumption on the orders of the singularities. Moreover, it is a quadratic form with respect to the intersection form: $Q(u+v) = Q(u)+Q(v)+\langle u,v\rangle$ [Ati71; Joh80; Zor08, Appendix C]. Now, we define the *spin parity* of M to be the value assumed most often by Q, that is, the Arf invariant of Q. Another equivalent definition uses a maximal symplectic subset $(u_\alpha,v_\alpha)_{\alpha\in\mathcal{A}}$ of $H_1(M\setminus\Sigma;\mathbb{Z}/2\,\mathbb{Z})$ and puts $\operatorname{Arf}(Q) = \sum_{\alpha\in\mathcal{A}} Q(u_\alpha)Q(v_\alpha)$, which does not depend on the choice of maximal symplectic subset.

The third and last invariant is called *regularity* and is only defined for a specific finite list of strata of the moduli space of half-translation surfaces (see below). There is no known geometric interpretation of this invariant, although an algebraic interpretation is known in terms of the dimension of $H^0(M, (\phi)_0/3)$, where $(\phi)_0$ is the zero divisor of the quadratic differential ϕ defining M.

We can now state the complete classifications, first for Abelian strata and then for quadratic strata.

Theorem 1.2.6 ([KZ03]). *The following is the classification of the connected components of strata of Abelian differentials on genus-g Riemann surfaces.*

- *In genus 1, the only stratum is $\mathcal{H}(0)$. It is nonempty, connected and hyperelliptic.*
- *In genus 2, the only two strata are $\mathcal{H}(2)$ and $\mathcal{H}(1, 1)$. Both of them are nonempty, connected and hyperelliptic.*
- *In genus 3, the strata $\mathcal{H}(4)$ and $\mathcal{H}(2, 2)$ have two components. One of them is hyperelliptic and the other corresponds to odd spin structures. They are identified by the labels* hyp *and* odd*, respectively. Every other stratum is nonempty and connected.*
- *Assume now that $g \geq 4$.*
  - *The stratum $\mathcal{H}(2g-2)$ has three components. One of them is hyperelliptic and the other two correspond to even and odd spin structures. They are identified by the labels* hyp, even *and* odd*, respectively.*
  - *The stratum $\mathcal{H}(g-1, g-1)$ can have either two or three components depending on the parity of g. If g is odd, it has three components. One of them is hyperelliptic and the other two are defined by even or odd spin structures. They are identified by the labels* hyp, even *and* odd*, respectively. If g is even, it has two components. One of them is hyperelliptic and the other is not. They are identified by the labels* hyp *and* nonhyp*, respectively.*
  - *All other strata of the form $\mathcal{H}(2m_1, \ldots, 2m_n)$ have two connected components, which correspond to even and odd spin structures. They are identified by the labels* even *and* odd*, respectively.*
  - *Every other stratum is nonempty and connected.*

Theorem 1.2.7 ([Lan08; CM14]). *The following is the classification of the connected components of strata of quadratic differentials on genus-g Riemann surfaces.*

- *In genus 0, every stratum is nonempty and connected.*
- *In genus 1, the strata $\mathbb{Q}(0)$ and $\mathbb{Q}(1, -1)$ are empty. Every other stratum is nonempty and connected.*
- *In genus 2, the strata $\mathbb{Q}(4)$ and $\mathbb{Q}(3, 1)$ are empty.*
- *In genus 3, the strata $\mathbb{Q}(9, -1)$, $\mathbb{Q}(6, 3, -1)$ and $\mathbb{Q}(3, 3, 3, -1)$ have two connected components, called "regular" and "irregular". They are identified by the labels* reg *and* irr*, respectively.*
- *In genus 4, the strata $\mathbb{Q}(6, 6)$, $\mathbb{Q}(6, 3, 3)$ and $\mathbb{Q}(3, 3, 3, 3)$ have three connected components. One of them is hyperelliptic and the others are not and are called "regular" and "irregular". They*

are identified by the labels hyp, reg *and* irr, *respectively. Moreover, the strata* $\mathbb{Q}(12)$ *and* $\mathbb{Q}(9, 3)$ *have two connected components, also called "regular" and "irregular". They are identified by the labels* reg *and* irr, *respectively.*

- *Assume now that* $g \geq 2$. *The strata of the form* $\mathbb{Q}(4k + 2, 4j + 2)$, $\mathbb{Q}(4k + 2, 2j - 1, 2j - 1)$ *and* $\mathbb{Q}(2k - 1, 2k - 1, 2j - 1, 2j - 1)$, *for* $k, j \geq 0$ *not covered in the previous list, have two components. One of them is hyperelliptic and the other is not. They are identified by the labels* hyp *and* nonhyp, *respectively. Every other stratum is nonempty and connected.*

The strata of the moduli space of quadratic differentials that have an unusual number of connected components, that is, $\mathbb{Q}(9, -1)$, $\mathbb{Q}(6, 3, -1)$ and $\mathbb{Q}(3, 3, 3, -1)$ in genus 3, and $\mathbb{Q}(6, 6)$, $\mathbb{Q}(6, 3, 3)$, $\mathbb{Q}(3, 3, 3, 3)$, $\mathbb{Q}(12)$ and $\mathbb{Q}(9, 3)$ in genus 4, are called *exceptional strata*. Moreover, the strata of the moduli space of flat surfaces having only one singularity are called *minimal strata*. Finally, the strata of the form $\mathcal{H}(1, 1, \ldots, 1)$ or $\mathbb{Q}(-1, -1, \ldots, -1, 1, 1, \ldots, 1)$ are called *principal strata*.

Coming back to the familiar examples, the surface on the left of Figure 1.2 belongs to $\mathcal{H}(4)^{\mathrm{odd}}$, as it does not belong to $\mathcal{H}(4)^{\mathrm{hyp}}$ since it does not possess a hyperelliptic symmetry, while the surface on the right belongs to $\mathcal{H}(3, 1)$, which is connected. On the other hand, the surfaces shown in Figure 1.3 belong to $\mathbb{Q}(8)$ and $\mathbb{Q}(2, 2, 1, -1)$, from left to right, which are connected.

Period coordinates, the Gauss–Manin connection and the Masur–Veech measure

Period coordinates provide a way to parametrise the Teichmüller and moduli space of flat surfaces in terms of cohomology groups. We will describe them in detail for Abelian differentials and we will then succinctly discuss the case of quadratic differentials.

Let $\mathcal{S}$ be a stratum of the Teichmüller space of translation surfaces. Let $M_0 \in \mathcal{S}$ and let ω_0 be an Abelian defining the flat structure of M_0. We denote the set of singularities of ω_0 by $\Sigma_0 \subseteq M_0$. By writing M_0 as a collection of polygons P_0, we have that the homology group $H_1(M_0, \Sigma_0; \mathbb{C})$ is generated by the relative cycles induced by orienting each side of P_0. Now, we can consider each of these sides as an element of $\mathbb{C}$ and define an open set $M_0 \ni U \subseteq \mathcal{S}$ by perturbing each side inside an open set of $\mathbb{C}$ (in such a way that identified sides remain parallel and still have the same length). Let $M \in U$ be given an Abelian ω, whose set of singularities is $\Sigma \subseteq M$, and also by a collection of polygons P obtained by perturbing P_0. The homology group $H_1(M, \Sigma; \mathbb{C})$ is generated by the relative cycles induced by orienting each side of P. Since we produced P by perturbing P_0, we obtain an isomorphism between $H_1(M, \Sigma; \mathbb{C})$ and $H_1(M_0, \Sigma_0; \mathbb{C})$, and therefore an isomorphism between the cohomology groups $H^1(M, \Sigma; \mathbb{C})$ and $H^1(M_0, \Sigma_0; \mathbb{C})$. Moreover, ω can be regarded as a cohomology class in $H^1(M, \Sigma; \mathbb{C})$ via the formula $\theta \mapsto \int_\theta \omega$ and, therefore, as a cohomology class in $H^1(M_0, \Sigma_0; \mathbb{C})$. The map $\mathcal{P}_U : U \to H^1(M_0, \Sigma_0; \mathbb{C})$ defined in this way is a homeomorphism and is called a *period map*. Furthermore, this way to identify nearby (co)homology groups is called the *Gauss–Manin con-*

nection. The cohomology group $H^1(M_0, \Sigma_0; \mathbb{C})$ is an n-dimensional complex vector space, with $n = 2g + |\Sigma_0| - 1$, and, thus, the pairs $(U, \mathcal{P}_U)$ constructed in this way are explicit charts for the stratum $\mathcal{S}$ called *period coordinates* [Mas82]. Moreover, by considering isomorphisms $H^1(M_0, \Sigma_0; \mathbb{C}) \simeq \mathbb{C}^n$, it is easy to see that the transition maps for these coordinate charts are affine volume-preserving maps of $\mathbb{C}^n$ as they belong to $\mathrm{SL}(n, \mathbb{Z}) \subseteq \mathrm{GL}(n, \mathbb{C})$. Therefore, it is possible to define a "Lebesgue" measure $\lambda_{\mathcal{S}}$ on $\mathcal{S}$ as the pullback of the Lebesgue measure on $\mathbb{C}^n$, which is well-defined up to normalization. This measure is called the *Masur–Veech measure* and the usual normalization requires the volume of $H^1(M, \Sigma; \mathbb{C})/H^1(M, \Sigma; \mathbb{Z} \oplus i\mathbb{Z})$ to be 1. It also descends to the moduli space of translation surfaces into a "Lebesgue" measure also called the *Masur–Veech measure*. As the Lebesgue measure on $\mathbb{C}^n$, the Masur–Veech measure has infinite total volume on each stratum, but it can be disintegrated into a *finite* measure on the subspace $\mathcal{H}^{(1)}(\kappa) \subseteq \mathcal{H}(\kappa)$ consisting on area-one translation surfaces [Mas82; Vee82].

In the case of half-translation surfaces, similar coordinates can be defined, but it is not possible to use the quadratic differential directly: it does not define a cohomology class as it cannot be integrated along homology classes in a coherent way. However, the orientable double cover construction provides a way to solve this issue. Indeed, given a half-translation surface M_0 endowed with a quadratic differential ϕ_0, let $\widetilde{M}_0$ be its orientable double cover which is endowed with an Abelian differential ω_0. Let $\iota\colon \widetilde{M}_0 \to \widetilde{M}_0$ be the involution exchanging the fibres of the cover. Since $\iota^*\omega_0 = -\omega_0$ by construction, the zeroes $\widetilde{\Sigma}_0$ of ω_0 are fixed by ι. Thus, ι induces a well-defined map on the relative cohomology group $H^1(\widetilde{M}_0, \widetilde{\Sigma}_0; \mathbb{C})$. Since ι is an involution, $H^1(\widetilde{M}_0, \widetilde{\Sigma}_0; \mathbb{C})$ can be split into a direct sum of the invariant and anti-invariant subspaces for ι. We have that ω_0 belongs to the anti-invariant part. By identifying nearby fibres in an way analogous to the case of translation surfaces (that is, using the Gauss–Manin connection), the anti-invariant part of $H^1(\widetilde{M}_0, \widetilde{\Sigma}_0; \mathbb{C})$ provides natural coordinates for the Teichmüller space of quadratic differentials also called *period coordinates* [Mas82]. An analogous "Lebesgue" measure, also called the *Masur–Veech measure*, can be defined in this case. This measure descends to the moduli space of quadratic differentials, as expected, and it can also be desintegrated into a finite measure on the subspace $\mathbb{Q}^{(1)}(\kappa) \subseteq \mathbb{Q}(\kappa)$ consisting of area-one half-translation surfaces [Vee90].

1.2.3 Dynamics of flat surfaces

We have now defined an interesting group action and invariant measure on the moduli space of flat surfaces. Therefore, we can study their dynamical properties. A fundamental result is the following, going back to the seminal works of Masur and Veech:

Theorem 1.2.8 ([Mas82; Vee82; Vee86]). *For any stratum of the moduli space of flat surfaces, the Teichmüller geodesic flow (and, therefore, the* $\mathrm{SL}(2, \mathbb{R})$*-action) is ergodic with respect to the Masur–Veech measure.*

Stronger quantitative results are now known [AGY06; AR12; AG13]:

Theorem 1.2.9. *For any stratum of the moduli space of flat surfaces, the Teichmüller geodesic flow is exponentially mixing with respect to the Masur–Veech measure.*

We say that a connected subspace $\mathcal{M}$ of a stratum of the moduli space of flat surfaces is an *affine invariant submanifold* if it is an immersed submanifold of a stratum locally defined in period coordinates by linear equations with real coefficients and zero constant terms. We say that a set of the form $\overline{\mathrm{GL}^+(2,\mathbb{R})\cdot M}$ for some flat surface M is an *orbit closure*. A very deep result relates these two notions:

Theorem 1.2.10 ([Fil16; EM18; EMM15]). *Any orbit closure is an algebraic variety defined over $\overline{\mathbb{Q}}$. Moreover, it is an affine invariant submanifold.*

The Hodge bundle, the Kontsevich–Zorich cocycle and Lyapunov exponents

The Hodge bundle is the fibre bundle whose base space is a stratum of the moduli space of flat surfaces and its fibre is a suitable (co)homology group. The Kontsevich–Zorich cocycle is the dynamical cocycle over the Hodge bundle induced by the $\mathrm{SL}(2,\mathbb{R})$-action or by the Teichmüller geodesic flow. Therefore, it encodes the homological part of the action of $\mathrm{SL}(2,\mathbb{R})$ or of its diagonal subgroup on the moduli space of flat surfaces. It was originally studied by Kontsevich and Zorich [KZ97; Kon97].

In the case of translation surfaces, the (co)homology group is usually chosen to be $H_1(S;\mathbb{R})$, where S is the underlying topological surface, and nearby fibres are identified using the Gauss–Manin connection. In the case of half-translation surfaces, however, two fundamentally different versions of the Hodge bundle exist, using either the invariant or anti-invariant parts of the homology of its orientable double cover. We will now state the precise definitions, starting with the case of translation surface and then discussing the case of half-translation surfaces.

Let $\mathcal{TH}(\kappa)$ be a stratum of the Teichmüller space of translation surfaces and let $\mathcal{TH}^{(1)}(\kappa)$ be its subset of area-one surfaces and S be the underlying topological surface. We define the trivial vector bundle $\widehat{H(\kappa)} = \mathcal{TH}^{(1)}(\kappa)\times H_1(S,\mathbb{R})$ and a trivial dynamical cocycle over $\widehat{H(\kappa)}$ by $\widehat{G_g}(M,v) = (g\cdot M, v)$. In this expression, g can be taken either as an arbitrary element of $\mathrm{SL}(2,\mathbb{R})$ or as $\mathrm{diag}(e^t, e^{-t})$, depending on whether we are interested on the $\mathrm{SL}(2,\mathbb{R})$-action or on the Teichmüller geodesic flow.

Now, we can project this cocycle down to $H(\kappa) = \widehat{H(\kappa)}/\mathrm{Mod}(S)$, where $\mathrm{Mod}(S)$ acts on *both* factors. We obtain a "cocycle" $G_g\colon H(\kappa)\to H(\kappa)$ which is called the *Kontsevich–Zorich cocycle*. Strictly speaking, G_g is not a dynamical cocycle: $H(\kappa)$ is not a vector bundle because the action of $\mathrm{Mod}(S)$ on $\mathcal{TH}(\kappa)$ is not free. Nevertheless, we can replace $\mathrm{Mod}(S)$ by a finite-index subgroup that acts freely on $\mathcal{TH}(\kappa)$ to obtain a finite cover of $H(\kappa)$ which is a vector bundle. We will usually ignore these orbifold issues in the rest of this thesis.

For the case of a half-translation surface, the homological action is defined on the homology group of its double cover. Since this group splits into an invariant and anti-invariant part which are symplectic and symplectically orthogonal, this splitting induces two “disjoint” homological actions usually referred as the “plus” and “minus” subbundles.

Remark 1.2.11. It is also possible to define the homological action on the homology group of the half-translation surface directly. This is equivalent to the “plus” part of the homology of the double cover. However, as shown by our discussion on period coordinates, the “minus” part should also be taken into account.

Remark 1.2.12. There are several versions of the Kontsevich–Zorich cocycle in the literature. We have defined it as the homological action of either the Teichmüller geodesic flow or the $\mathrm{SL}(2,\mathbb{R})$-action on $H_1(S;\mathbb{R})$. Instead of using real coefficients for the homology group, one can use integer coefficients, complex coefficients, etc. Moreover, one can take marked points into account, either by considering the absolute homology group of the punctured surface or the homology group of the surface relative to them.

Furthermore, it can also be defined as the homological action of all paths in the moduli space space of translation surfaces starting at a given point (instead of just the paths obtained by following the $\mathrm{SL}(2,\mathbb{R})$-orbits). This version of the Kontsevich–Zorich cocycle is related to the *monodromy group*, which can be defined as the group spanned by all possible symplectic automorphisms produced by the homological action of all possible closed paths at a given surface. In the case of square-tiled surfaces (defined in Section 1.2.4), however, another definition of monodromy group is used by following $\mathrm{SL}(2,\mathbb{R})$-orbits instead of arbitrary paths.

The multiplicative ergodic theorem, originally proven by Oseledets (also spelled Oseledec) [Ose68], allows us to define Lyapunov exponents for a dynamical cocycle satisfying some integrability conditions. The general theory of Lyapunov exponents implies that the Lyapunov exponents of any symplectic cocycle of rank $2g$ are of the form:

$$\lambda_1 \geq \lambda_2 \geq \cdots \geq \lambda_g \geq 0 \geq -\lambda_g \geq \cdots \geq -\lambda_2 \geq -\lambda_1.$$

The Lyapunov exponents of the Kontsevich–Zorich cocycle have been a subject of much recent interest. In the case of translation surfaces, it is not difficult to prove that $\lambda_1 = 1$, by using the so-called *tautological plane*. Moreover, Veech proved that $1 = \lambda_1 > \lambda_2$ [Vee86]. The Kontsevich–Zorich conjecture states that the spectrum is simple: all the Lyapunov exponents are distinct. The first step towards this conjecture was done by Forni [For02], who proved that $\lambda_g > 0$ and, thus, established the nonuniform hyperbolicity of the Kontsevich–Zorich cocycle with respect to the Masur–Veech measures. The full conjecture was proved by Avila and Viana [AV07a; AV07b] using properties of the Rauzy–Veech groups (defined in Chapter 2). For half-translation surfaces, the conjecture states that the “plus” and “minus” Lyapunov spectra (that is, the Lyapunov spectrum of the cocycle restricted to the invariant and anti-invariant parts of the homology of the double cover, respectively) are simple. These exponents are usually written as

$$\lambda_1^+ \geq \lambda_2^+ \geq \cdots \geq \lambda_g^+ \geq 0 \geq -\lambda_g^+ \geq \cdots \geq -\lambda_2^+ \geq -\lambda_1^+.$$

and

$$\lambda_1^- \geq \lambda_2^- \geq \cdots \geq \lambda_{\tilde{g}-g}^- \geq 0 \geq -\lambda_{\tilde{g}-g}^- \geq \cdots \geq -\lambda_2^- \geq -\lambda_1^-.$$

where g is the genus of the original half-translation surface and $\tilde{g}$ is the genus of its double cover. It is known that $\lambda_1^- = 1$ as the tautological plane belongs to the anti-invariant part. Furthermore, Forni's result [For02] implies that $1 = \lambda_1^- > \lambda_2^-$ and that $1 > \lambda_1^+$. Moreover, by the work of Treviño [Tre13] we have that $\lambda_g^+ > 0$ and $\lambda_{\tilde{g}-g}^- > 0$, so the Kontsevich–Zorich cocycle is also nonuniformly hyperbolic in this case. We will show in Chapter 3 that the Lyapunov spectra are simple provided some conditions on the orders of the singularities are satisfied.

Finally, there is remarkable formula by Eskin, Kontsevich and Zorich that allows one to easily compute the *sum* of the Lyapunov exponents of a given component of a stratum of the moduli space of quadratic differentials [EKZ14].

Monodromy groups

Let $\mathcal{M}$ be an affine invariant submanifold. An $\mathrm{SL}(2,\mathbb{R})$-invariant subbundle E of the Hodge bundle is a subbundle for which $g \cdot E_X = E_{g \cdot X}$ for every $X \in \mathcal{M}$ and $g \in \mathrm{SL}(2,\mathbb{R})$. A flat subbundle E is a subbundle which is flat for the Gauss–Manin connection. Observe that a flat subbundle is necessarily $\mathrm{SL}(2,\mathbb{R})$-invariant, since if the curvature vanishes then the parallel transport is done along $\mathrm{SL}(2,\mathbb{R})$-orbits in the "obvious" way. The converse is not true in general: the flatness condition requires no curvature in every possible direction, including those which are not reachable by the $\mathrm{SL}(2,\mathbb{R})$-action. The classification of the $\mathrm{SL}(2,\mathbb{R})$-invariant subbundles which are not flat is known [EFW18].

The Hodge bundle can be decomposed into irreducible pieces and monodromy groups can be defined for such pieces: each monodromy group is spanned by the homological action on such piece of the Hodge bundle via the Kontsevich–Zorich cocycle.

Monodromy groups are discrete subgroups, but we are usually interested in their Zariski-closures. The following theorem poses constraints on the possible Zariski-closures of monodromy groups.

Theorem 1.2.13 ([Fil17]). *Let E be a strongly irreducible $\mathrm{SL}(2,\mathbb{R})$-invariant subbundle of the Hodge bundle over some affine invariant submanifold $\mathcal{M}$. Then, the Zariski-closure of the monodromy group, at the level of real Lie algebra representations and up to compact factors, belongs to the following list*:

(*i*) $\mathfrak{sp}(2g,\mathbb{R})$ *in the standard representation*;
(*ii*) $\mathfrak{su}(p,q)$ *in the standard representation*;
(*iii*) $\mathfrak{su}(p,1)$ *in an exterior power representation*;
(*iv*) $\mathfrak{so}^*(2d)$ *in the standard representation*; *or*
(*v*) $\mathfrak{so}_{\mathbb{R}}(n,2)$ *in a spin representation.*

Nevertheless, it is not known whether every Lie algebra representation in this list is realisable as a monodromy group [Fil17, Question 1.5]. Indeed, it is well-known that every group in the first item is realisable. The groups in the second item were shown to be realisable by Avila, Matheus and Yoccoz [AMY19]. Moreover, the group $\mathrm{SO}^*(6)$ in its standard representation (which coincides with $\mathrm{SU}(3, 1)$ in its second exterior power representation) is also realisable by the work of Filip, Forni and Matheus [FFM18]. We will show that $\mathrm{SO}^*(2d)$ is realisable for each $11 \le d \le 299$ such that $d = 3 \bmod 8$, except possibly for $d = 35$ and $d = 203$, in Chapter 4.

Under stronger hypothesis, one also has a classification at the level of Lie group representations:

Theorem 1.2.14 ([Fil17]). *Let E be a strongly irreducible flat subbundle of the Hodge bundle over some affine invariant submanifold $\mathcal{M}$. Then, the presence of zero Lyapunov exponents implies that the Zariski-closure of the monodromy group has at most one noncompact factor, which, up to finite-index, is equal at the level of Lie group representations to*:

1. $\mathrm{SU}(p, q)$ *in the standard representation*;
2. $\mathrm{SU}(p, 1)$ *in any exterior power representation*; *or*
3. $\mathrm{SO}^*(2d)$ *in the standard representation for some odd d.*

The Rauzy–Veech algorithm

The Rauzy–Veech algorithm, originally studied by Rauzy [Rau77; Rau79] and Veech [Vee82], is an explicit way to code the Teichmüller geodesic flow and its homological action. Indeed, it provides concrete bases of homology and elements of the mapping class group which can be used to express the Kontsevich–Zorich cocycle as explicit matrices. This algorithm has been fruitfully used to obtain many interesting results. In particular, it was used by Avila and Viana to prove the Kontsevich–Zorich conjecture stating the simplicity of the Lyapunov spectra of all components of all strata of the moduli space of translation surfaces [AV07a; AV07b].

The algorithm is not defined directly on the moduli space of quadratic differentials, but on a finite cover consisting on surfaces with a specific choice of a horizontal line segment L stemming rightwards from one of its marked points. A rigorous way to define this cover is to use Veech's *zippered rectangles* construction [Vee82]. However, we will define a slightly less general version below. We refer the reader to the lecture notes by Yoccoz for more details [Yoc10].

Assume that a flat surface M is given by a generalised permutation representative π (of type (ℓ, m)) and a suspension datum $\{\zeta_\alpha\}_{\alpha \in \mathcal{A}}$ defining a suitable polygon. In this case, there is a natural choice of L: the line segment joining 0 and $\mathrm{Re}(e_\pi)$. We can define a transversal T as such flat surfaces whose horizontal line segments have length exactly 1. The Teichmüller geodesic flow continuously increases the length of L. When the length becomes exactly $1 + \min(\mathrm{Re}(\zeta_{\pi(\ell)}, \zeta_{\pi(\ell+m)}))$, we apply a cut-and-paste operation and change the generalised permutation representative and the suspension datum in order to reduce the length of L to 1. This process is called the *Rauzy–Veech algorithm*. We will now describe the precise cut-and-paste operations and the way the generalised permutation and suspension datum change under them.

Depending on the suspension datum, there are either one or two possible generalised permutations that can arise from a given generalise permutation by this algorithm, as defined by Boissy and Lanneau [BL09]. We will start by defining these operations directly on the generalised permutations and we will then take the suspension datum into account.

If $\sigma(\ell) > \ell$, then $R_{\mathrm{t}}(\pi)$ is the type-(ℓ, m) generalised permutation defined as:

$$R_{\mathrm{t}}(\pi)(i) = \begin{cases} \pi(i) & i \leq \sigma(\ell) \\ \pi(\ell + m) & i = \sigma(\ell) + 1 \\ \pi(i - 1) & \text{otherwise;} \end{cases}$$

if $\sigma(\ell) < \ell$ and there exists duplicate letter in the bottom row of π which is not the last letter, then $R_{\mathrm{t}}(\pi)$ is the type-$(\ell + 1, m - 1)$ generalised permutation defined as:

$$R_{\mathrm{t}}(\pi)(i) = \begin{cases} \pi(i) & i < \sigma(\ell) \\ \pi(\ell + m) & i = \sigma(\ell) \\ \pi(i - 1) & \text{otherwise;} \end{cases}$$

and, in any other case, R_{t} is not defined on π. When a top operation is defined, we call $\pi(\ell)$ the *winner* and $\pi(\ell + m)$ the *loser* of the operation.

Similarly, if $\sigma(\ell + m) < \ell$, then $R_{\mathrm{b}}(\pi)$ is the type-(ℓ, m) generalised permutation defined as:

$$R_{\mathrm{b}}(\pi)(i) = \begin{cases} \pi(\ell) & i = \sigma(\ell + m) + 1 \\ \pi(i - 1) & \sigma(\ell + m) + 1 < i \leq \ell \\ \pi(i) & \text{otherwise;} \end{cases}$$

if $\sigma(\ell + m) > \ell$ and there exists duplicate letter in the top row of π which is not the last letter, then $R_{\mathrm{b}}(\pi)$ is the type-$(\ell - 1, m + 1)$ generalised permutation defined as:

$$R_{\mathrm{b}}(\pi)(i) = \begin{cases} \pi(i + 1) & \ell \leq i < \sigma(\ell + m) + 1 \\ \pi(\ell) & i = \sigma(\ell + m) - 1 \\ \pi(i) & \text{otherwise;} \end{cases}$$

and, in any other case, R_{b} is not defined on π. When a bottom operation is defined, we call $\pi(\ell + m)$ the *winner* and $\pi(\ell)$ the *loser* of the operation.

Observe that, if π is irreducible, then at least one of these operations is defined on π. Moreover, $R_{\mathrm{t}}(\pi)$, $R_{\mathrm{b}}(\pi)$ are also irreducible if they are defined.

Once a Rauzy–Veech operation is clear from context, we will denote the winner of the operation by α_{w} and its loser by α_{l}.

Remark 1.2.15. In the case of (genuine) permutations, we always have that $\sigma(\ell) > \ell$ and that $\sigma(\ell + m) < \ell$, so only one case can occur in the definition of R_{t} and R_{b}.

Now, let $\zeta = \{\zeta_\alpha\}_{\alpha\in\mathcal{A}}$ be a suspension datum for π. We say that $\zeta = \{\zeta_\alpha\}_{\alpha\in\mathcal{A}}$ is of *type* 0 if $\zeta_{\pi(\ell)} > \zeta_{\pi(\ell+m)}$ and that it is of *type* 1 if $\zeta_{\pi(\ell)} < \zeta_{\pi(\ell+m)}$. We define the pair $(\pi', \zeta') = R(\pi, \zeta)$ resulting from the Rauzy–Veech algorithm by:

- $R(\pi, \zeta) = (R_{\mathrm{t}}(\pi), \zeta')$ with $\zeta'_\alpha = \zeta_\alpha$ if $\alpha \neq \pi(\ell)$ and $\zeta'_{\pi(\ell)} = \zeta_{\pi(\ell)} - \zeta_{\pi(\ell+m)}$ if (π, ζ) has type 0, and
- $R(\pi, \zeta) = (R_{\mathrm{b}}(\pi), \zeta')$ with $\zeta'_\alpha = \zeta_\alpha$ if $\alpha \neq \pi(\ell + m)$ and $\zeta'_{\pi(\ell+m)} = \zeta_{\pi(\ell+m)} - \zeta_{\pi(\ell)}$ if (π, ζ) has type 1.

Remark 1.2.16. If $\zeta_{\pi(\ell)} = \zeta_{\pi(\ell+m)}$, then the Rauzy–Veech algorithm is not defined. This is not a problem since, for any fixed generalised permutation, the subset of suspension data satisfying this condition is a Lebesgue-negligible set.

Observe that the matrix in $\mathrm{SL}(\mathbb{C}^{\mathcal{A}})$ mapping ζ' to ζ can be expressed as $\mathrm{Id} + E_{\alpha_{\mathrm{l}}\alpha_{\mathrm{w}}}$ (where ζ is considered as a *row* vector).

Rauzy diagrams

The Rauzy–Veech operations allow us to define a directed graph on the set of generalised permutations. These graphs are called *Rauzy diagrams* and their sets of vertices are called *Rauzy classes*. In other words, two generalised permutations belong to the same Rauzy class if they can be joined by a sequence of Rauzy–Veech operations. The ergodicity of the Teichmüller geodesic flow implies that the Rauzy diagrams are strongly connected. Moreover, any directed path in a Rauzy class can be realised by Rauzy–Veech operations on flat surfaces.

Rauzy classes are in a one-to-one correspondence with the connected components of the moduli space of flat surfaces with a suitable choice of horizontal line segment. When this line segment is forgotten, the correspondence becomes finite-to-one.

A cycle at a fixed generalised permutation can be, thus, though of as an almost-closed trajectory of the Teichmüller geodesic flow. The homological action of such cycles, defined by using the Gauss–Manin–Kontsevich–Zorich cocycle, can be regarded as "almost" being the homological action of the Teichmüller geodesic flow. Nevertheless, the subset of $\mathrm{SL}(2, \mathbb{R})$ stabilising a surface is trivial for generic surfaces, so the word "almost" cannot be dropped: typical orbits of the Teichmüller geodesic flow are never closed.

The groups that arise as all possible homological actions of such almost-closed trajectories are called *Rauzy–Veech groups*. In the case of translation surface, they are defined and completely classified in Chapter 2, while in the case of half-translation surfaces they are defined and partially classified in Chapter 3.

Homology groups and permutation representatives

Let π be an irreducible generalised permutation (possibly a genuine permutation). There exist natural maps

$$H_1(M_\pi \setminus \Sigma_\pi) \to H_1(M_\pi) \to H_1(M_\pi, \Sigma_\pi), \tag{$*$}$$

the former being surjective and the latter injective. We refer the reader to the survey by Viana [Via06] and the lecture notes by Yoccoz [Yoc10] for more details.

We will first assume that π is a (genuine) permutation. Let $\{\theta_\alpha\}_{\alpha\in\mathcal{A}}$ be the curves in M_π joining the midpoints of the sides labelled α in P_α, oriented upwards. These curves generate $\pi_1(M_\pi \setminus \Sigma_\pi)$ and induce a basis $\{e_\alpha\}_{\alpha\in\mathcal{A}}$ of $H_1(M_\pi \setminus \Sigma_\pi)$. Moreover, the relative cycles $\{f_\alpha\}_{\alpha\in\mathcal{A}}$ induced by the α-sides of P_π are a basis of $H_1(M_\pi, \Sigma_\pi)$ which is dual to $\{e_\alpha\}_{\alpha\in\mathcal{A}}$: $\langle e_\alpha, f_\beta\rangle = \delta_{\alpha\beta}$ for each $\alpha, \beta \in \mathcal{A}$. The symplectic form Ω_π given by the intersection of the curves $\{\theta_\alpha\}_{\alpha\in\mathcal{A}}$ can be explicitly expressed in purely combinatorial terms:

$$(\Omega_\pi)_{\alpha\beta} = \begin{cases} +1 & \pi_{\mathrm{t}}(\alpha) < \pi_{\mathrm{t}}(\beta) \text{ and } \pi_{\mathrm{b}}(\alpha) > \pi_{\mathrm{b}}(\beta) \\ -1 & \pi_{\mathrm{t}}(\alpha) > \pi_{\mathrm{t}}(\beta) \text{ and } \pi_{\mathrm{b}}(\alpha) < \pi_{\mathrm{b}}(\beta) \\ 0 & \text{otherwise.} \end{cases}$$

The image of $H_1(M_\pi)$ inside $H_1(M_\pi, \Sigma_\pi)$, which is naturally isomorphic to $H_1(M_\pi)$, is the set $H(\pi)$ spanned by $\{\Omega_\pi f_\alpha\}_{\alpha\in\mathcal{A}}$ (acting on column vectors). There is a natural nondegenerate symplectic form on $H(\pi)$ defined by $\langle \Omega_\pi v, \Omega_\pi w\rangle = v^\top \Omega_\pi w$.

The kernel of the first map in $(*)$ is equal to $\ker\Omega_\pi$. Indeed, they have the same rank and every element of $\ker\Omega_\pi$ is mapped to $0 \in H_1(M_\pi)$ as the intersection form on $H_1(M_\pi)$ is nondegenerate. Therefore, $H_1(M_\pi \setminus \Sigma_\pi)/\ker\Omega_\pi$ and $H_1(M_\pi)$ are naturally isomorphic. Moreover, the symplectic form Ω_π descends to $H_1(M_\pi \setminus \Sigma_\pi)/\ker\Omega_\pi$ into a nondegenerate symplectic form.

We obtain, thus, two symplectic spaces which are naturally isomorphic to $H_1(M_\pi)$. The symplectic isomorphism obtained by composing the maps in $(*)$ can be described by the formula $\Omega_\pi f_\alpha \mapsto [e_\alpha]$ for each $\alpha \in \mathcal{A}$. We will now show that the monodromy actions on these isomorphic spaces are dual to each other.

A closed path at $M_\pi \setminus \Sigma_\pi$ inside the moduli space of flat surfaces induces a matrix A in terms of the basis $\{e_\alpha\}_{\alpha\in\mathcal{A}}$ (acting on *row* vectors) and a matrix B in terms of the basis $\{f_\alpha\}_{\alpha\in\mathcal{A}}$ (acting on *column* vectors) by using the Kontsevich–Zorich cocycle with the appropriate fibre. Since A is symplectic, it preserves $\ker\Omega_\pi$. Thus, A induces a well-defined action on the quotient $H_1(M_\pi \setminus \Sigma_\pi)/\ker\Omega_\pi$. Moreover, the matrix B preserves $H(\pi)$ since the set of closed curves is preserved by the Gauss–Manin connection.

By duality and symplecticity, we have that $\langle e_\alpha A, Bf_\beta\rangle = \delta_{\alpha\beta}$ for each $\alpha, \beta \in \mathcal{A}$. Moreover, since $e_\alpha A = \sum_\gamma A_{\alpha\gamma} e_\gamma$ and $Bf_\beta = \sum_\gamma B_{\beta\gamma} f_\gamma$ we have that

$$\langle e_\alpha A, Bf_\beta\rangle = \sum_\gamma A_{\alpha\gamma} B_{\gamma\alpha} = \delta_{\alpha\beta}$$

which shows that $B = A^{-1}$. Since $A\Omega_\pi A^\intercal = \Omega_\pi$, we obtain that $B\Omega_\pi = A^{-1}\Omega_\pi A^{-\intercal}A^\intercal = \Omega_\pi A^\intercal$ and, thus, that

$$B\Omega_\pi f_\alpha = \Omega_\pi A^\intercal f_\alpha \mapsto [e_\alpha A].$$

Therefore, the (right-)action on $H_1(M_\pi \setminus \Sigma_\pi)/\ker\Omega_\pi$ and the (left-)action on $H_1(M_\pi, \Sigma_\pi)$ are mutually dual. Thus, the monodromy group in "absolute homology" (or its subgroups) can be obtained in three equivalent ways: using $H_1(M_\pi)$ as a fibre, using $H_1(M_\pi \setminus \Sigma_\pi)$ as a fibre and then modding out by $\ker\Omega_\pi$, and using $H_1(M_\pi, \Sigma_\pi)$ and then restricting to $H(\pi)$. In our discussion about Rauzy–Veech groups in Chapter 2, we will use the second fibre since this choice makes the definition of the Kontsevich–Zorich matrices (defined in the next section) more natural.

Similar facts hold for half-translation surfaces. Indeed, assume now that π is a strict generalised permutation. We will start by describing the homological action on the surface M_π (which corresponds to the invariant or "plus" subbundle) and we will then discuss homological action on the double cover $\widetilde{M}_\pi$.

Let $\{\theta_\alpha\}_{\alpha\in\mathcal{A}}$ be the curve joining the midpoints of the sides M_π, oriented left to right, right to left and upwards for sides that are both in the top row, both in the bottom row, and in both rows, respectively. These curves generate $\pi_1(M_\pi \setminus \Sigma_\pi)$ and induce a natural basis $\{e_\alpha\}_{\alpha\in\mathcal{A}}$ of $H_1(M_\pi \setminus \Sigma_\pi)$. The relative cycles $\{f_\alpha\}_{\alpha\in\mathcal{A}}$ induced by the α-sides of P_π are a basis of $H_1(M_\pi, \Sigma_\pi)$ which is dual to $\{e_\alpha\}_{\alpha\in\mathcal{A}}$. The symplectic form Ω_π given by the intersection of the curves $\{\theta_\alpha\}_{\alpha\in\mathcal{A}}$ is given by

$$(\Omega_\pi)_{\alpha\beta} = \begin{cases} +1 & i_\alpha < i_\beta \le \ell \text{ and } j_\alpha > j_\beta > \ell \\ +1 & i_\alpha < i_\beta < j_\alpha < j_\beta \le \ell \\ +1 & i_\beta < i_\alpha < j_\beta \le \ell < j_\alpha \\ +1 & j_\alpha > j_\beta > i_\alpha > \ell \text{ and } i_\alpha > i_\beta \\ -1 & i_\beta < i_\alpha \le \ell \text{ and } j_\beta > j_\alpha > \ell \\ -1 & i_\beta < i_\alpha < j_\beta < j_\alpha \le \ell \\ -1 & i_\alpha < i_\beta < j_\alpha \le \ell < j_\beta \\ -1 & j_\beta > j_\alpha > i_\beta > \ell \text{ and } i_\beta > i_\alpha \\ 0 & \text{otherwise.} \end{cases}$$

where the integers $i_\alpha, j_\alpha \in \{1, \dots, 2d\}$ are defined by $\pi^{-1}(\alpha) = \{i_\alpha, j_\alpha\}$, with $i_\alpha < j_\alpha$.

As in the Abelian case, a homological action can be defined on the three spaces in $(*)$. The previous discussion carries over almost word-for-word: the image $V^+(\pi)$ of the second map inside $H_1(M_\pi, \Sigma_\pi)$ is spanned by $\{\Omega_\pi f_\alpha\}_{\alpha\in\mathcal{A}}$, the spaces $V^+(\pi)$ and $H_1(M_\pi \setminus \Sigma_\pi)/\ker\Omega_\pi$ are both naturally isomorphic to $H_1(M_\pi)$, and the resulting homological actions induced by the monodromy group are mutually dual.

Now, we will present the double cover construction of M_π in a combinatorial way. Define

$P_\pi^0 = P_\pi$ and let P_π^1 be a translation of $-P_\pi$ which is disjoint from P_π. Let $\widetilde{P}_\pi = P_\pi^0 \cup P_\pi^1$ and $p\colon \widetilde{P}_\pi \to P_\pi$ be the natural two-to-one covering between $\widetilde{P}_\pi$ and P_π. Let $\iota'\colon \widetilde{P}_\pi \to \widetilde{P}_\pi$ be the involution exchanging P_π^0 and P_π^1 by translations and central symmetries. We label the sides of $\widetilde{P}_\pi$ with the alphabet $\mathcal{A} \times \{0, 1\}$ so that the following conditions hold for any $\alpha \in \mathcal{A}$ and $\varepsilon \in \{0, 1\}$:

- p maps an (α, ε)-side of $\widetilde{P}_\pi$ to an α-side of P_π;
- ι' maps an (α, ε)-side of $\widetilde{P}_\pi$ to an $(\alpha, 1-\varepsilon)$-side of $\widetilde{P}_\pi$;
- P_π^0 contains both $(\alpha, 0)$-sides of $\widetilde{P}_\pi$ if and only if α occurs in both rows of π.

These conditions ensure that, when identifying equally-labelled sides, one obtains a valid translation surface $\widetilde{M}_\pi$, equipped with a well-defined involution $\iota\colon \widetilde{M}_\pi \to \widetilde{M}_\pi$ induced by ι', whose quotient $\widetilde{M}_\pi/\iota$ is M_π. Let $\widetilde{\Sigma}_\pi = p^{-1}(\Sigma_\pi)$. See Figure 3.1 for an example of this construction.

The involution ι induces the splitting $H_1(\widetilde{M}_\pi \setminus \widetilde{\Sigma}_\pi) = H^+(\pi) \oplus H^-(\pi)$ into invariant and anti-invariant parts. We will not use a explicit basis for $H^-(\pi)$, although we will use a basis for a subspace of $H^-(\pi)$ in Chapter 3.

Homological action of the Rauzy–Veech algorithm

The homological action of the Rauzy–Veech induction algorithm can be described as a change of basis. As usual, we will start with the case of translation surfaces and then discuss the case of half-translation surfaces.

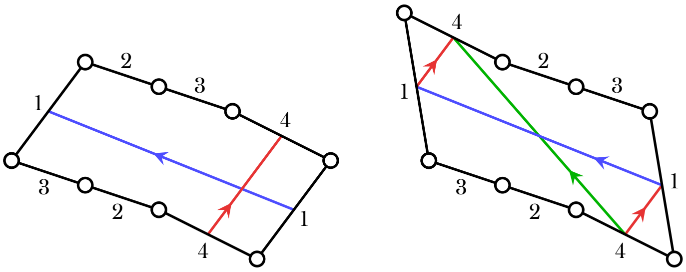
Let π be a (genuine) permutation. Let $\{e_\alpha\}_{\alpha\in\mathcal{A}}$ be the basis of $H_1(M_\pi \setminus \Sigma_\pi)$ defined in the last section and let $\{e'_\alpha\}_{\alpha\in\mathcal{A}}$ be the analogous basis for $H_1(M_{\pi'} \setminus \Sigma_{\pi'})$, where $\gamma = \pi \to \pi'$ is an arrow of a Rauzy diagram. We define the *Kontsevich–Zorich matrix* indexed by $\mathcal{A} \times \mathcal{A}$ as $B_\gamma = \mathrm{Id} + E_{\alpha_\mathrm{l}\alpha_\mathrm{w}} \in \mathrm{SL}(\mathbb{Z}^{\mathcal{A}})$, where Id is the identity matrix and $E_{\alpha_\mathrm{l}\alpha_\mathrm{w}}$ has only one nonzero coefficient, equal to 1, at position $\alpha_\mathrm{l}\alpha_\mathrm{w}$. The matrix B_γ corresponds to the homological action $H_1(M_{\pi'} \setminus \Sigma_{\pi'}) \to H_1(M_\pi \setminus \Sigma_\pi)$ with respect to the bases $\{e'_\alpha\}_{\alpha\in\mathcal{A}}$ and $\{e_\alpha\}_{\alpha\in\mathcal{A}}$, as shown by Figure 1.4.

In the case of strict generalised permutations, we will only describe the homological action explicitly for the "plus" part (although we will analyse this action on a subspace of the "minus" part in Chapter 3). Let $\gamma = \pi \to \pi'$ be an arrow of a Rauzy diagram. We define the *Kontsevich–Zorich matrix* indexed by $\mathcal{A} \times \mathcal{A}$ as

$$B_\gamma = \begin{cases} \mathrm{Id} + E_{\alpha_\mathrm{l}\alpha_\mathrm{w}} & \langle e_{\alpha_\mathrm{l}}, e_{\alpha_\mathrm{w}}\rangle \neq 0 \\ \mathrm{Id} - E_{\alpha_\mathrm{l}\alpha_\mathrm{w}} - 2E_{\alpha_\mathrm{l}\alpha_\mathrm{l}} & \langle e_{\alpha_\mathrm{l}}, e_{\alpha_\mathrm{w}}\rangle = 0 \end{cases} \in \mathrm{SL}(\mathbb{Z}^{\mathcal{A}}).$$

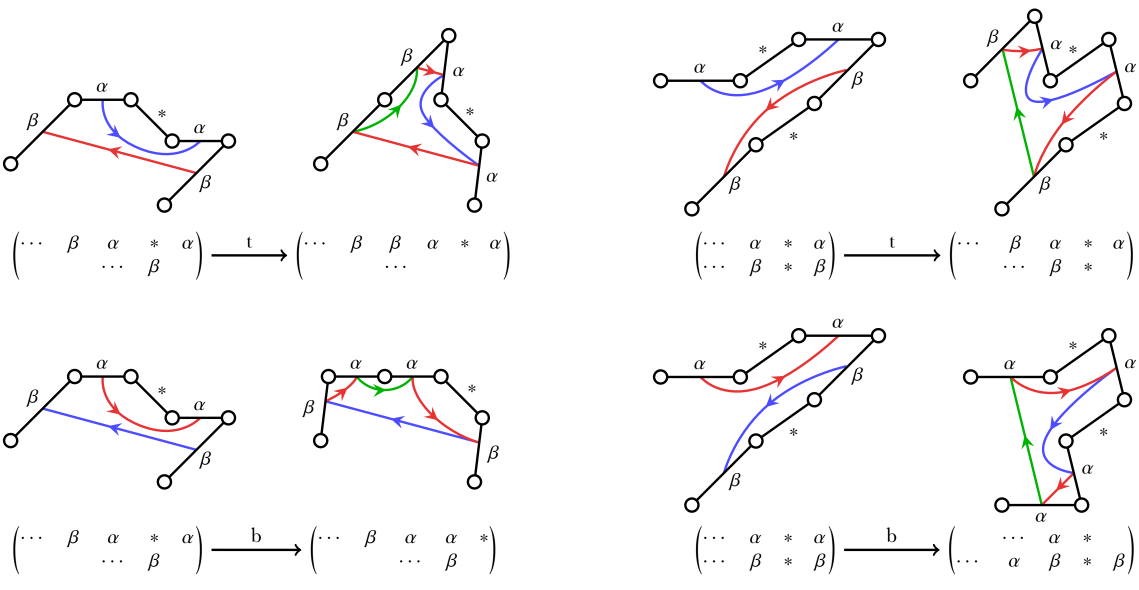
The matrix B_γ corresponds to the homological action $H_1(M_{\pi'} \setminus \Sigma_{\pi'}) \to H_1(M_\pi \setminus \Sigma_\pi)$ with respect to the bases $\{e'_\alpha\}_{\alpha\in\mathcal{A}}$ and $\{e_\alpha\}_{\alpha\in\mathcal{A}}$. See Figure 1.5 for some examples of cases that do not show up in the Abelian case.

Remark 1.2.17. Observe that the homological action of the "plus" part of the homology for the case of generalised permutations is more general than the case of (genuine) permutations.



$$\begin{pmatrix} 1 & 2 & 3 & 4 \\ 3 & 2 & 4 & 1 \end{pmatrix} \xrightarrow{\quad \text{b} \quad} \begin{pmatrix} 1 & 4 & 2 & 3 \\ 3 & 2 & 4 & 1 \end{pmatrix}$$

Figure 1.4: Computation of the Kontsevich–Zorich matrices as a change of basis in homology for the Abelian case. The red line represents e_{α_1}, the blue line, $e_{\alpha_{\mathrm{w}}} = e'_{\alpha_{\mathrm{w}}}$ and the green line, e'_{α_1}. Observe that $e_{\alpha_1} = e'_{\alpha_1} + e_{\alpha_{\mathrm{w}}}$.



$$\begin{pmatrix} \cdots & \beta & \alpha & * & \alpha \\ & & \cdots & \beta & \end{pmatrix} \xrightarrow{\ \text{t}\ } \begin{pmatrix} \cdots & \beta & \beta & \alpha & * & \alpha \\ & & \cdots & & & \end{pmatrix} \qquad \begin{pmatrix} \cdots & \alpha & * & \alpha \\ \cdots & \beta & * & \beta \end{pmatrix} \xrightarrow{\ \text{t}\ } \begin{pmatrix} \cdots & \beta & \alpha & * & \alpha \\ & \cdots & \beta & * & \end{pmatrix}$$

$$\begin{pmatrix} \cdots & \beta & \alpha & * & \alpha \\ & & \cdots & \beta & \end{pmatrix} \xrightarrow{\ \text{b}\ } \begin{pmatrix} \cdots & \beta & \alpha & \alpha & * \\ & & \cdots & \beta & \end{pmatrix} \qquad \begin{pmatrix} \cdots & \alpha & * & \alpha \\ \cdots & \beta & * & \beta \end{pmatrix} \xrightarrow{\ \text{b}\ } \begin{pmatrix} & \cdots & \alpha & * & \\ \cdots & \alpha & \beta & * & \beta \end{pmatrix}$$

Figure 1.5: Computation of the Kontsevich–Zorich matrices as a change of basis in homology when $\langle e_{\alpha_1}, e_{\alpha_{\mathrm{w}}} \rangle = 0$. The red lines represent e_{α_1}, the blue lines, $e_{\alpha_{\mathrm{w}}} = e'_{\alpha_{\mathrm{w}}}$ and the green lines, e'_{α_1}. Observe that, in all of these cases, $e_{\alpha_1} + e'_{\alpha_1} + e_{\alpha_{\mathrm{w}}} = 0$, so the map $H_1(M_{\pi'} \setminus \Sigma_{\pi'}) \to H_1(M_\pi \setminus \Sigma_\pi)$ is represented in these bases as $\mathrm{Id} - E_{\alpha_1\alpha_{\mathrm{w}}} - 2E_{\alpha_1\alpha_1}$.

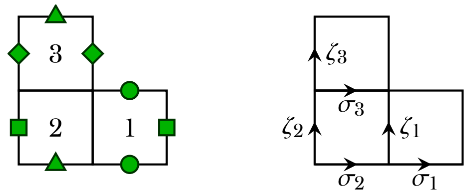


Figure 1.6: An example of an L-shaped square-tiled surface associated with the permutations $h = (2, 1)(3)$ and $v = (1)(2, 3)$ (left). The relative cycles σ_n and ζ_n (right).

Indeed, $\langle \theta_{\alpha_1}, \theta_{\alpha_w} \rangle$ is never 0, so we recover the original formula. Nevertheless, we chose to present these two cases separately since they differ in a fundamental aspect: for translation surfaces, the matrices expressing the homological action are always positive and coincide with the action on suspension data, while for half-translation surfaces this is not true. This means that we cannot use the homological action on half-translation surfaces as a coding for the Teichmüller geodesic flow, a fact that does hold for translation surfaces. To analyse the Lyapunov spectra of quadratic differentials in Chapter 3, thus, we need to use another coding (due to Avila and Resende [AR12]) on the base, while we use the aforementioned matrices on the fibre.

1.2.4 Square-tiled surfaces

Square-tiled surface are very particular examples of translation surfaces. They are interesting because, while they can be described in simple combinatorial terms, they also exhibit a rich behaviour. We will start by presenting two equivalent definitions.

A *square-tiled surface* (also called an *origami*) is a translation surface M such that there exists a finite cover $\pi\colon M \to \mathbb{T}^2 = \mathbb{C}/(\mathbb{Z}\oplus i\mathbb{Z})$ branched only at $0 \in \mathbb{T}^2$ whose Abelian differential ω satisfies $\omega = \pi^*(dz)$. Equivalently, it is a pair of permutations $(h, v) \in \mathrm{Sym}(\mathrm{Sq}(M)) \times \mathrm{Sym}(\mathrm{Sq}(M))$ acting transitively on $\mathrm{Sq}(M)$, where $\mathrm{Sq}(M)$ is a finite set. See Figure 1.6 for an example. This notion was introduced by Thurston [Thu88] and first studied from the dynamical point of view by Gutkin [Gut84], Veech [Vee87] and Gutkin and Judge [GJ96; GJ00].

The equivalence between these two definitions can be proved as follows: the permutations h, v can be obtained from the original definition as the deck transformations induced respectively by the curves $t \mapsto (t, 0)$ and $t \mapsto (0, t)$, with $t \in [0, 1]$, and the set of squares $\mathrm{Sq}(M)$ can be defined to be the set of connected components of $\pi^{-1}((0, 1) \times (0, 1))$. Conversely, we can glue squares horizontally using h and vertically using v and define ω to be the pullback of dz in each square to obtain a square-tiled surface as in the original definition.

We say that square-tiled surface M is *reduced* if the covering map π cannot be factored through another nontrivial covering of the torus. In this case, the elements $g \in \mathrm{SL}(2, \mathbb{R})$ such that $g \cdot M$ is a square-tiled surface are exactly $\mathrm{SL}(2, \mathbb{Z})$. It is often the case that we study the action of $\mathrm{SL}(2, \mathbb{Z})$ on M instead of the entire $\mathrm{SL}(2, \mathbb{R})$-action, since square-tiled surfaces can be represented in purely combinatorial terms. The *Veech group* of M, usually denoted $\mathrm{SL}(M)$, is the subgroup of $\mathrm{SL}(2, \mathbb{Z})$ stabilising M [Vee89; Vee91]. It is always a finite-index subgroup of $\mathrm{SL}(2,\mathbb{Z})$ and its index coincides with the cardinality of $\mathrm{SL}(2,\mathbb{Z})\cdot M$. Every square-tiled surface that we will consider is reduced.

Homology groups

Let S be the underlying topological surface of M and let Σ be the set of vertices of the squares of M. For each $n \in \mathrm{Sq}(M)$, we let $\sigma_n \in H_1(S,\Sigma)$ be the bottom horizontal side of n, oriented rightwards, and let $\zeta_n \in H_1(S,\Sigma)$ be the left vertical side of n, oriented upwards. See Figure 1.6. The group $H_1(S,\Sigma)$ can be thought of as the quotient of the group of formal sums on $\{\sigma_n,\zeta_n\}_{n\in\mathrm{Sq}(M)}$ by the relations $\sigma_n+\zeta_{h(n)}-\sigma_{v(n)}-\zeta_n$ for each $n\in\mathrm{Sq}(M)$.

The homology group $H_1(S;\mathbb{R})$ admits a splitting $H_1^{\mathrm{st}}(S)\oplus H_1^{(0)}(S)$ into symplectic and mutually symplectically orthogonal subspaces. The subspace $H_1^{\mathrm{st}}(S)$ is two-dimensional and is usually called the *tautological plane*. It is spanned by $\sum_{n\in\mathrm{Sq}(M)}\sigma_n$ and $\sum_{n\in\mathrm{Sq}(M)}\zeta_n$. The subspace $H_1^{(0)}(S)$ consists of the *zero-holonomy cycles*, that is, the cycles c such that $\int_c\omega=0$.

Automorphisms and affine homeomorphisms

A square-tiled surface may also have nontrivial automorphisms. In this case, the $\mathrm{SL}(2,\mathbb{Z})$-action does not immediately induce a homological action on the Hodge bundle. Indeed, automorphisms are precisely the reason why orbit closures are, in general, orbifolds and not manifolds. More precisely, we define an *affine homeomorphism* as an orientation-preserving homeomorphism of M whose local expressions (with respect to the translation atlas) are affine maps of $\mathbb{R}^2$. We denote the group of affine homeomorphisms by $\mathrm{Aff}(M)$. We may extract the linear part of an affine homeomorphism to get a surjective homomorphism $\mathrm{Aff}(M)\to\mathrm{SL}(M)$. The kernel of this homomorphism is the group $\mathrm{Aut}(M)$ of automorphisms of M. This can be encoded in the form of a short exact sequence:

$$1\to\mathrm{Aut}(M)\to\mathrm{Aff}(M)\to\mathrm{SL}(M)\to 1.$$

In other words, $\mathrm{Aut}(M)$ is precisely the subgroup of $\mathrm{Mod}(S)$ stabilizing a lift of M to the Teichmüller space of translation surfaces. In this sense, it measures to which extent the $\mathrm{Mod}(S)$-action fails to be free at M. Automorphisms can also be defined combinatorically: they are the elements of $\mathrm{Sym}(\mathrm{Sq}(M))$ that commute with both h and v.

It is well-known that if M has only one singularity, then it has no nontrivial automorphisms:

Proposition 1.2.18. *Let M be a square-tiled surface belonging to a minimal stratum $\mathcal{H}(2g-2)$. Then,* $\mathrm{Aut}(M)=\{\mathrm{Id}\}$.

Proof. Assume that there exists $\psi\in\mathrm{Aut}(M)\setminus\{\mathrm{Id}\}$. Let $p\in M$ be the only singularity of M. Since ψ is nontrivial, p is the unique fixed point of ψ. Moreover, ψ has finite order, say $k\geq 2$. Let $N=M/\langle\psi\rangle$. Then, the covering map $\pi\colon M\to N$ induces a normal cover of degree k which is not ramified outside p and is ramified of order k at p.

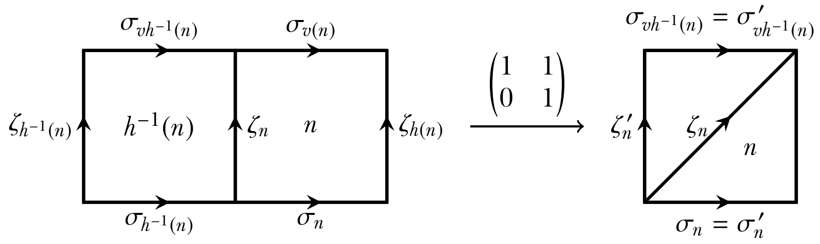


Figure 1.7: The action of T_{h} on σ_n and ζ_n. Observe that $\zeta_n = \zeta'_n + \sigma'_{vh^{-1}(n)}$.

Let $* \in N \setminus \{q\}$, where $q = \pi(p)$, be a basepoint. Then, the covering map π is given by a homomorphism $f\colon \pi_1(N \setminus \{q\}) \to \mathbb{Z}/k\mathbb{Z}$. In this language, a small loop $\gamma \in \pi_1(N \setminus \{q\}, *)$ around q based at $*$ is a product of commutators in $\pi_1(N \setminus \{q\})$, so that $f(\gamma) = 1$. This contradicts the fact that π is ramified at p with degree $k \geq 2$. □

Let $\rho\colon \mathrm{Aff}(M) \to \mathrm{Sp}(H_1(S;\mathbb{R}))$ be the representation induced by the homological action of $\mathrm{Aff}(M)$. By restricting this representation to an invariant subspace, we obtain a monodromy representation of a subbundle of the Hodge bundle. We define the *monodromy group* of this subbundle to be the image of this representation.

Remark 1.2.19. While a typical translation surface is not stabilised by any nontrivial element of $\mathrm{SL}(2,\mathbb{R})$, a square-tiled surface is stabilised by a "very large" subgroup (a finite-index subgroup of $\mathrm{SL}(2,\mathbb{Z})$). Thus, it is reasonable to define its monodromy group by following the $\mathrm{SL}(2,\mathbb{R})$-orbits, which is something that cannot be done for a generic surface.

The group $\rho(\mathrm{Aff}(M))$ preserves the splitting $H_1(S;\mathbb{R}) = H_1^{\mathrm{st}}(S) \oplus H_1^{(0)}(S)$. Moreover, the tautological plane $H_1^{\mathrm{st}}(S)$ is irreducible and its monodromy group is a finite-index subgroup of $\mathrm{SL}(2,\mathbb{Z}) = \mathrm{Sp}(2,\mathbb{Z})$ which can be identified with $\mathrm{SL}(M)$. The subspace $H_1^{(0)}(S)$ is in general reducible. Therefore, understanding monodromy groups of subbundles of the Hodge bundle means understanding the irreducible pieces of $H_1^{(0)}(S)$ and the way $\rho(\mathrm{Aff}(M))$ acts on them.

The $\mathrm{SL}(2,\mathbb{Z})$-action on homology

The group $\mathrm{SL}(2,\mathbb{Z})$ is generated by the following two matrices corresponding to horizontal and vertical shears, respectively:

$$T_{\mathrm{h}} = \begin{pmatrix}1 & 1\\ 0 & 1\end{pmatrix} \quad \text{and} \quad T_{\mathrm{v}} = \begin{pmatrix}1 & 0\\ 1 & 1\end{pmatrix}.$$

Thus, it is enough to study the action of these two matrices on $H_1(S;\mathbb{R})$ to understand the entirety of the $\mathrm{SL}(2,\mathbb{Z})$-action. The action of T and S on $H_1(S,\Sigma;\mathbb{R})$ can be explicitly described as follows:

$$(T_{\mathrm{h}})_*\sigma_n = \sigma'_n \quad \text{and} \quad (T_{\mathrm{h}})_*\zeta_n = \zeta'_n + \sigma'_{vh^{-1}(n)},$$

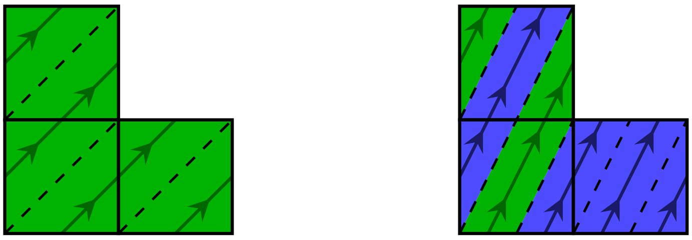


Figure 1.8: The cylinder decompositions for the directions (1, 1) and (1, 2), and their associated waist curves in the L-shaped surface.

$$(T_\mathrm{v})_*\sigma_n = \sigma_n'' + \zeta_{hv^{-1}(n)}'' \quad \text{and} \quad (T_\mathrm{v})_*\zeta_n = \zeta_n'',$$

where $\{\sigma_n', \zeta_n'\}_{n\in\mathrm{Sq}(T_\mathrm{h}\cdot M)}$ and $\{\sigma_n'', \zeta_n''\}_{n\in\mathrm{Sq}(T_\mathrm{v}\cdot M)}$ are the generating cycles for $T_\mathrm{h}\cdot M$ and $T_\mathrm{v}\cdot M$, respectively. See Figure 1.7.

Cylinder decompositions and Dehn multi twists

While the action of the matrices T_h and T_v on $H_1(S, \Sigma)$ is, in principle, enough to understand the $\mathrm{SL}(2, \mathbb{Z})$-action on $H_1(S; \mathbb{R})$, in practice it is usually more convenient to exploit the geometry of M by using cylinder decompositions and Dehn multi twists.

We say that a *cylinder* is a maximal collection of parallel closed geodesics bounded by saddle connections. Given any rational direction $(p, q) \in \mathbb{Q}^2$, a square-tiled surface M decomposes into a finite collection of cylinders with disjoint interiors that are parallel to the segment joining $(0, 0)$ and (p, q). The *waist curve* of a cylinder is the closed curve going through the middle of the cylinder. Let $\{C_i\}_{i=1}^k$ be the cylinder decomposition of M induced by a given rational direction (p, q) and let $w_i \in H_1(S)$ be the waist curve of C_i. Assume that $\ell_1 \geq \ell_i$ for every $i = 1, \ldots, k$. We have that the ratios of lengths $\{\ell_1/\ell_i\}_{i=1}^k$ are commensurable, so there exists a minimal integer $N \geq 1$ such that $N\ell_1/\ell_i$ is an integer for every $i = 1, \ldots, k$. The *Dehn multi twist* along (p, q) is the affine homeomorphism obtained by performing $N\ell_1/\ell_i$ Dehn twists along the waist curve w_i for each $i = 1, \ldots, k$. This homeomorphism is well-defined since $\langle w_i, w_j\rangle = 0$ for each $i \neq j$ and, thus, the Dehn twists along the waist curves commute with one another. Moreover, it is indeed an affine homeomorphism because it can be obtained by cutting and pasting along the diagonal of each cylinder and then shearing in the direction (p, q).

Constraints for monodromy groups

Let $G = \mathrm{Aut}(M)$. The vector space $H_1(S; \mathbb{R})$ has a structure of a G-module induced by the representation $G \to \mathrm{Sp}(H_1(S; \mathbb{R}))$. Since G is a finite group, it admits finitely many irreducible representations over $\mathbb{R}$ which we denote $\mathrm{Irr}_\mathbb{R}(G)$. The G-module $H_1(S; \mathbb{R})$ can be decomposed as a direct sum of irreducible representations. That is:

$$H_1(S;\mathbb{R}) = \bigoplus_{\alpha\in \mathrm{Irr}_{\mathbb{R}}(G)} V_\alpha^{\oplus n_\alpha},$$

where each V_α is an irreducible subspace of $H_1(S;\mathbb{R})$ on which G acts as the representation α.

We can collect the same G-irreducible representations into the so-called *isotypical components*. That is, let $W_\alpha = V_\alpha^{\oplus n_\alpha}$ and then:

$$H_1(S;\mathbb{R}) = \bigoplus_{\alpha\in \mathrm{Irr}_{\mathbb{R}}(G)} W_\alpha.$$

The group $\rho(\mathrm{Aff}(M))$ does not, a priori, respect this decomposition because a general affine homeomorphism may not commute with every automorphism. However, since G is a finite group, there exists a finite-index subgroup of $\mathrm{Aff}_*(M) \le \mathrm{Aff}(M)$ whose every element commutes with every element of G. Replacing $\mathrm{Aff}(M)$ by some finite-index subgroup preserves connected component of the identity of the Zariski-closure of the resulting monodromy group.

Given an irreducible representation α of G, we may define an associative division algebra D_α: the centraliser of $\alpha(G)$ inside $\mathrm{End}_{\mathbb{R}}(V_\alpha)$. Up to isomorphism, there are three associative real division algebras:

- $D_\alpha \simeq \mathbb{R}$, and α is said to be real;
- $D_\alpha \simeq \mathbb{C}$, and α is said to be complex; or
- $D_\alpha \simeq \mathbb{H}$, and α is said to be quaternionic.

The following theorem [MYZ14, Section 3.7; MYZ16] relates these cases to constraints for monodromy groups:

Theorem 1.2.20. *The Zariski-closure of the group* $\rho(\mathrm{Aff}_*(M))|_{W_\alpha}$ *is contained in*:

- $\mathrm{Sp}(2g_\alpha, \mathbb{R})$ *if* α *is real*;
- $\mathrm{SU}(p_\alpha, q_\alpha)$ *if* α *is complex*; *or*
- $\mathrm{SO}^*(2d_\alpha)$ *if* α *is quaternionic.*

Chapter 2

Classification of Rauzy–Veech groups of Abelian differentials

The purpose of this chapter is to classify the Rauzy–Veech groups of all connected components of all strata of Abelian differentials. It is an adapted version of the article "Classification of Rauzy–Veech groups: proof of the Zorich conjecture" [Gut19b]. Throughout the entire chapter, we will only work with (genuine) permutations instead of generalised permutations.

2.1 Introduction

The Kontsevich–Zorich conjecture states the simplicity of the Lyapunov spectra of almost all translation flows with respect to the Masur–Veech measures. The main ingredient of Avila and Viana's proof is the fact that Rauzy–Veech groups are pinching and twisting, which is implied by Zariski-density by the work of Benoist [Ben97]. However, the converse is not true: there are known examples of pinching and twisting groups with small Zariski closure [AMY18, Appendix A].

The pioneering work of Avila, Matheus and Yoccoz [AMY18] shows that this conjecture holds for the particular case of hyperelliptic Rauzy–Veech groups. Their methods also laid the groundwork for further results in this direction.

The main theorem of this chapter is the following:

Theorem 2.1.1. *At the level of absolute homology, the Rauzy–Veech group of any connected component of any stratum of the moduli space of genus-g translation surfaces is an explicit finite-index subgroup of* $\mathrm{Sp}(2g, \mathbb{Z})$. *More precisely*:

- *For hyperelliptic connected components, it is the group preserving a specific finite set modulo two* [*AMY18, Theorem 2.9*]. *Equivalently, at the level of mapping classes, it is the group commuting with the hyperelliptic involution* [*AMY18, Section 4.2.2*].
- *For spin connected components, it is the preimage, under the modulo-two reduction, of an orthogonal group. In other words, it is the group preserving a specific quadratic form modulo two.*

- *Otherwise, it is the entire ambient symplectic group.*

This theorem is proved in several steps throughout the chapter. A more precise version of our results is presented in Theorem 2.6.8.

Remark 2.1.2. The Rauzy–Veech groups of a spin connected component is, *a posteriori*, a maximal subgroup of $\mathrm{Sp}(2g, \mathbb{Z})$ in the sense that any group properly containing it is $\mathrm{Sp}(2g, \mathbb{Z})$ [BGP14, Theorem 3][1]. Its index is $2^{g-1}(2^g + 1)$ or $2^{g-1}(2^g - 1)$, depending on the spin parity. Since $2^{g-1}(2^g - 1)$ does not divide $2^{g-1}(2^g + 1)$ for $g \geq 2$, any subgroup of $\mathrm{Sp}(2g, \mathbb{Z})$ containing the Rauzy–Veech group of both spin connected components is $\mathrm{Sp}(2g, \mathbb{Z})$. We use these facts several times in order to completely classify the Rauzy–Veech groups.

We obtain a similar result for the monodromy group, since it contains the Rauzy–Veech group, commutes with the hyperelliptic involution at the level of mapping classes for hyperelliptic connected components, and preserves the same quadratic form for spin connected components. The Zariski-density was already known for these groups [Fil17, Corollary 1.7].

Corollary 2.1.3. *At the level of absolute homology, the monodromy group of any connected component of any stratum of the moduli space of genus-g translation surfaces is an explicit finite-index subgroup of* $\mathrm{Sp}(2g, \mathbb{Z})$ *which coincides with the Rauzy–Veech group.*

To prove Theorem 2.1.1, we reduce the general conjecture to the case of minimal strata. Indeed, we show in Lemma 2.6.5 that the Rauzy–Veech group of any connected component of any stratum contains the Rauzy–Veech group of specific connected components of minimal strata of surfaces of the same genus. For the case of minimal strata, we prove the following theorem:

Theorem 2.1.4. *The Rauzy–Veech group of a nonhyperelliptic connected component of a minimal stratum of the moduli space of genus-g translation surfaces is the preimage of the orthogonal group* $\mathrm{O}(Q) \subseteq \mathrm{Sp}(2g, \mathbb{Z}/2\mathbb{Z})$ *by the modulo-two reduction* $\mathrm{Sp}(2g, \mathbb{Z}) \to \mathrm{Sp}(2g, \mathbb{Z}/2\mathbb{Z})$*, where* Q *is the quadratic form on modulo-two homology defining the spin parity.*

An immediate corollary is that these groups have finite index in $\mathrm{Sp}(2g, \mathbb{Z})$. The proof of Theorem 2.1.4 is elementary and is divided into two parts. First, we show that the level-two congruence subgroup of $\mathrm{Sp}(2g, \mathbb{Z})$ (that is, the kernel of the modulo-two reduction) is contained in such Rauzy–Veech groups in Proposition 2.3.6 and Proposition 2.3.14. Then, we show that the modulo-two reduction is surjective onto $\mathrm{O}(Q)$ in Proposition 2.4.8. Both parts of the proof rely on constructing explicit sets of generators inductively, first for the level-two congruence subgroup and then for $\mathrm{O}(Q)$.

Our methods also allow us to describe the Rauzy–Veech groups of the nonhyperelliptic connected components of $\mathcal{H}(g-1, g-1)$ for $g \geq 3$ at the level of relative homology explicitly. Indeed, we show in Theorem 2.5.1 that they can be written as suitable subgroups of semi-direct products.

[1] This result uses the classification of finite simple groups.

2.2 Rauzy–Veech groups

Although we already introduced the Rauzy–Veech groups in the previous chapter, will now give a more precise definition of Rauzy–Veech groups for translation surfaces.

2.2.1 Rauzy–Veech groups in homology

Let $\mathcal{R}$ be a Rauzy diagram. We consider an undirected version $\widetilde{\mathcal{R}}$ of $\mathcal{R}$: for each arrow $\gamma = \pi \to \pi'$ we add a reversed arrow $\gamma^{-1} = \pi' \to \pi$. To each arrow γ^{-1} we associate the matrix $B_{\gamma^{-1}} = B_\gamma^{-1}$, where B_γ is the Kontsevich–Zorich matrix of γ. Now consider a walk $\gamma = \gamma_1\gamma_2 \cdots \gamma_n$ in $\widetilde{\mathcal{R}}$ starting at π and ending at π'. We define $B_\gamma = B_{\gamma_n}B_{\gamma_{n-1}} \cdots B_{\gamma_1} \in \mathrm{SL}(\mathbb{Z}^{\mathcal{A}})$, which satisfies $\Omega_{\pi'} = B_\gamma \Omega_\pi B_\gamma^\intercal$. In particular, if $\pi' = \pi$ (that is, if γ is a cycle), one has that B_γ (acting on *row* vectors) belongs to $\mathrm{Sp}(\Omega_\pi, \mathbb{Z})$.

The Rauzy–Veech group of π is the group generated by matrices of this form:

Definition 2.2.1. Let $\mathcal{R}$ be a Rauzy diagram and let $\pi \in \mathcal{R}$ be a fixed vertex. We define the *Rauzy–Veech group* $\mathrm{RV}(\pi)$ of π as the set of matrices in $\mathrm{Sp}(\Omega_\pi, \mathbb{Z})$ of the form B_γ, where γ is a cycle on $\widetilde{\mathcal{R}}$ with endpoints at π. We will always consider the action of $\mathrm{RV}(\pi)$ on row vectors unless explicitly stated otherwise.

Observe if π, π' are vertices of the same Rauzy class $\mathcal{R}$, then $\mathrm{RV}(\pi)$ and $\mathrm{RV}(\pi')$ are isomorphic, so we can define the Rauzy–Veech group of a Rauzy class. Indeed, if γ is any walk joining π and π', then the conjugation by B_γ is an isomorphism between $\mathrm{Sp}(\Omega_\pi, \mathbb{Z})$ and $\mathrm{Sp}(\Omega_{\pi'}, \mathbb{Z})$ and between $\mathrm{RV}(\pi)$ and $\mathrm{RV}(\pi')$. This shows, in particular, that the Rauzy–Veech group of a Rauzy class has a well-defined index inside its ambient symplectic group.

Remark 2.2.2. One could also define Rauzy–Veech *monoids* by considering loops on the directed graph $\mathcal{R}$ instead of on the undirected graph $\widetilde{\mathcal{R}}$. Nevertheless, the group generated by this monoid coincides with our definition of Rauzy–Veech group, which is essentially a consequence of the fact that Rauzy diagrams are strongly connected. Nevertheless, classifying Rauzy–Veech monoids remains an open question.

2.2.2 Rauzy–Veech groups in homotopy

In this section we follow the general discussion about Rauzy diagrams and Dehn twists by Avila, Matheus and Yoccoz [AMY18, Section 4].

Consider a Rauzy diagram $\mathcal{R}$. For $\pi \in \mathcal{R}$, we denote the pure mapping class group of M_π relative to Σ_π by $\mathrm{Mod}(M_\pi, \Sigma_\pi)$. We equip M_π with a basepoint $*_\pi = 1/2 \in \mathbb{C}$ and we set $O_\pi = d/2 \in \mathbb{C}$. We denote by Σ_π^* the set consisting of O_π and the midpoints of the sides of P_π.

For each arrow $\gamma = \pi \to \pi'$ in $\mathcal{R}$ there exists a homeomorphism $H_\gamma\colon M_\pi \to M_{\pi'}$ respecting the naming of the sets of marked points $\Sigma_\pi, \Sigma_{\pi'}$ [AMY18, Section 4.1.2]. We denote by $[H_\gamma]$ its isotopy class from $(M_\pi, \Sigma_\pi \cup \Sigma_\pi^*)$ to $(M_\pi, \Sigma_{\pi'} \cup \Sigma_{\pi'}^*)$ relative to $\Sigma_\pi \cup \Sigma_\pi^*$. We also define $H_{\gamma^{-1}} = H_\gamma^{-1}$ for the arrow $\gamma^{-1} = \pi' \to \pi \in \widetilde{\mathcal{R}}$. Now consider a walk $\gamma = \gamma_1\gamma_2 \cdots \gamma_n$ in $\widetilde{\mathcal{R}}$

starting at π and ending at π'. Similarly, we define $[H_\gamma] = [H_{\gamma_n}][H_{\gamma_{n-1}}]\cdots[H_{\gamma_1}]$. We can then define the modular Rauzy–Veech group in an analogous way to the homological case:

Definition 2.2.3. Let $\mathcal{R}$ be a Rauzy diagram and let $\pi \in \mathcal{R}$ be a fixed vertex. We define the *modular Rauzy–Veech group* $\mathrm{MRV}(\pi)$ of π as the set of mapping classes in $\mathrm{Mod}(M_\pi, \Sigma_\pi)$ of the form $[H_\gamma]$, where γ is a cycle on $\widetilde{\mathcal{R}}$ with endpoints at π.

The action of the map H_γ on the fundamental groups, where $\gamma = \pi \to \pi'$ is an arrow in $\mathcal{R}$, can be made explicit. Indeed, let α_{w} and α_{l} be the winner and loser, respectively, of the Rauzy induction. The induced homomorphism $\pi_1(\gamma)\colon \pi_1(M_\pi \setminus \Sigma_\pi, *_\pi) \to \pi_1(M_{\pi'} \setminus \Sigma_{\pi'}, *_{\pi'})$ of H_γ on the fundamental groups is (up to homotopy): $\pi_1(\gamma)(\theta_\alpha) = \theta'_\alpha$ for every $\alpha \neq \alpha_{\mathrm{l}}$ and

$$\pi_1(\gamma)(\theta_{\alpha_{\mathrm{l}}}) = \begin{cases} \theta'_{\alpha_{\mathrm{l}}} \star (\theta'_{\alpha_{\mathrm{w}}})^{-1} & \text{if } \gamma \text{ is of top type} \\ (\theta'_{\alpha_{\mathrm{w}}})^{-1} \star \theta'_{\alpha_{\mathrm{l}}} & \text{if } \gamma \text{ is of bottom type,} \end{cases}$$

where $\{\theta_\alpha\}_{\alpha\in\mathcal{A}}$ and $\{\theta'_\alpha\}_{\alpha\in\mathcal{A}}$ are the distinguished curves in $M_\pi \setminus \Sigma_\pi$ and $M_{\pi'} \setminus \Sigma_{\pi'}$, respectively, joining the midpoints of the equally-labelled sides. The induced homomorphism of H_γ, written in terms of the homology classes of the curves $\{\theta_\alpha\}_{\alpha\in\mathcal{A}}$ and $\{\theta'_\alpha\}_{\alpha\in\mathcal{A}}$, is exactly $\mathrm{Id}-E_{\alpha_{\mathrm{l}}\alpha_{\mathrm{w}}} = B_\gamma^{-1}$. Therefore, the induced action of $\mathrm{MRV}(\pi)$ on homology is precisely $\mathrm{RV}(\pi)$. We will identify the matrix $B_\gamma = \mathrm{Id} + E_{\alpha_{\mathrm{l}}\alpha_{\mathrm{w}}}$ with a map $H_1(M_{\pi'} \setminus \Sigma_{\pi'}) \to H_1(M_\pi \setminus \Sigma_\pi)$ by using these bases.

Dehn twists

For each $\alpha \in \mathcal{A}$, we have that at least one of the Dehn twists along θ_α belongs to $\mathrm{MRV}(\pi)$. These Dehn twists will be useful to generate $\mathrm{RV}(\pi)$.

Lemma 2.2.4. *Let π be a vertex of a Rauzy diagram $\mathcal{R}$. Then, the left or right Dehn twist along θ_α belongs to* $\mathrm{MRV}(\pi)$ *for every $\alpha \in \mathcal{A}$.*

Proof. Fix $\alpha \in \mathcal{A}$ and let ξ_α be the isotopy class of θ_α. Let γ be a (possibly empty) walk on $\mathcal{R}$ starting at π ending on a vertex $\pi' = (\pi'_{\mathrm{t}}, \pi'_{\mathrm{b}})$ such that $\alpha = (\pi'_{\mathrm{t}})^{-1}(d)$ or $\alpha = (\pi'_{\mathrm{b}})^{-1}(d)$, and such that π' is the first vertex of γ with this property. Such a walk exists because Rauzy diagrams are strongly connected.

Let γ' be the pure cycle at π' having α as the winner. That is, all the arrows of γ are different and their winner is α. Assume that it is a pure cycle of top type. We have that $[H_{\gamma\gamma'\gamma^{-1}}]$ is equal to the left Dehn twist along θ_α, which we denote T_{ξ_α}. Indeed, let ξ'_α be the isotopy class of θ'_α. It is easy to see that $[H_{\gamma'}] = T_{\xi'_\alpha}$, since it is computed along a pure cycle [AMY18, Section 4.1.6]. We obtain that $[H_{\gamma\gamma'\gamma^{-1}}] = [H_\gamma^{-1}]T_{\xi'_\alpha}[H_\gamma] = T_{[H_\gamma^{-1}](\xi'_\alpha)}$, where the last equality is straightforward by definition of Dehn twist. Our hypothesis on γ and the previous discussion about the action of H_γ on the fundamental groups show that $[H_\gamma]$ maps ξ_α to ξ'_α, since α is not the loser of any arrow of γ. We conclude that $[H_{\gamma\gamma'\gamma^{-1}}] = T_{\xi_\alpha}$. If γ' is a pure cycle of bottom type, similar computations show that $[H_{\gamma\gamma'\gamma^{-1}}]$ is equal to the right Dehn twist along θ_α. □

Finally, the induced action $T_\alpha : H_1(M_\pi \setminus \Sigma_\pi) \to H_1(M_\pi \setminus \Sigma_\pi)$ of the (either left or right) Dehn twist T_{ξ_α} has a simple expression in the basis consisting of the homology classes of $\{\theta_\alpha\}_{\alpha \in \mathcal{A}}$. Indeed, the alternate form Ω_π is defined as the intersection form of such curves, so we have that $T_\alpha(u) = u + \langle [\theta_\alpha], u\rangle [\theta_\alpha]$ for any $u \in H_1(M_\pi \setminus \Sigma_\pi)$, where $[\theta_\alpha]$ is the homology class of θ_α and $\langle \cdot, \cdot \rangle$ is the bilinear form induced by Ω_π. Since we will use the basis $\{[\theta_\alpha]\}_{\alpha \in \mathcal{A}}$, we identify $[\theta_\alpha]$ with the canonical vector e_α whose only nonzero coordinate, equal to 1, is at position α.

2.2.3 General properties of Rauzy–Veech groups

In this section we state some general properties of Rauzy–Veech groups. Recall that we consider the action on *row* vectors. First, we state the definition of a symplectic transvection:

Definition 2.2.5. For a vector $v \in H_1(M_\pi \setminus \Sigma_\pi)$, we define the *symplectic transvection* T_v along v as:

$$T_v(u) = u + \langle v, u \rangle v.$$

Clearly, $T_v \in \mathrm{Sp}(\Omega_\pi, \mathbb{Z})$. Moreover, observe that $T_{-v} = T_v$ for any $v \in H_1(M_\pi \setminus \Sigma_\pi)$.

We have previously shown that $T_\alpha = T_{e_\alpha}$ for each $\alpha \in \mathcal{A}$. Conversely, for each primitive vector $v \in H_1(M_\pi \setminus \Sigma_\pi)$ we can choose a simple closed curve c on $M_\pi \setminus \Sigma_\pi$ whose homology class is v. If ξ is the isotopy class of c, then it is easy to see that the (either left or right) Dehn twist $T_\xi \in \mathrm{Mod}(M_\pi, \Sigma_\pi)$ acts in homology as T_v.

The following lemma allows us to construct more symplectic transvections from a set of generators.

Lemma 2.2.6. *Let $v, w \in H_1(M_\pi \setminus \Sigma_\pi)$ such that $\langle v, w \rangle = 1$. Then,*
- $T_w^{-1} T_v T_w = T_v T_w T_v^{-1} = T_{v+w}$;
- $T_w T_v T_w^{-1} = T_v^{-1} T_w T_v = T_{v-w}$.

Proof. It is a well-known fact that if $f \in \mathrm{Mod}(M_\pi, \Sigma_\pi)$, then $f T_\xi f^{-1} = T_{f(\xi)}$ [FM12, Fact 3.7]. The proof follows directly from this fact and the previous discussion, since the equality $\langle v, w \rangle = 1$ implies that v and w are primitive. □

We have that the group generated by symplectic transvections along canonical vectors is an invariant of the Rauzy class:

Lemma 2.2.7. *Let π, π' be vertices of a Rauzy diagram $\mathcal{R}$. For $\alpha \in \mathcal{A}$, define $T_{e_\alpha} \in \mathrm{RV}(\pi)$ and $T_{e'_\alpha} \in \mathrm{RV}(\pi')$ as the symplectic transvections along the homology classes of θ_α and θ'_α, respectively. Let $G \subseteq \mathrm{RV}(\pi)$ and $G' \subseteq \mathrm{RV}(\pi')$ be the subgroups generated by $\{T_{e_\alpha}\}_{\alpha \in \mathcal{A}}$ and $\{T_{e'_\alpha}\}_{\alpha \in \mathcal{A}}$, respectively. If γ is any walk in $\widetilde{\mathcal{R}}$ joining π and π', then the isomorphism $\mathrm{Sp}(\Omega_\pi, \mathbb{Z}) \to \mathrm{Sp}(\Omega_{\pi'}, \mathbb{Z})$ defined by $S \mapsto B_\gamma^{-1} S B_\gamma$ restricts to an isomorphism between G and G'.*

Proof. By induction, we can assume that γ is the arrow $\pi \to \pi'$. It is enough to show that, for each $\alpha \in \mathcal{A}$, there exists $S \in G$ such that $B_\gamma^{-1} S B_\gamma = T_{e'_\alpha}$. Observe that $B_\gamma^{-1} T_v B_\gamma = T_{v B_\gamma^{-1}}$ for any

$v \in H_1(M_\pi \setminus \Sigma_\pi)$. Therefore, if $\alpha \neq \alpha_1$, we can choose $S = T_{e_\alpha}$ and, if $\alpha = \alpha_1$, we can choose $S = T_{e_\alpha + e_{\alpha_{\mathrm{w}}}}$, which is equal to either $T_{e_{\alpha_{\mathrm{w}}}}^{-1} T_{e_\alpha} T_{e_{\alpha_{\mathrm{w}}}}$ or $T_{e_{\alpha_{\mathrm{w}}}} T_{e_\alpha} T_{e_{\alpha_{\mathrm{w}}}}^{-1}$ depending on the type of γ. □

We will end up showing that, in absolute homology, the group G in the previous lemma is the entire Rauzy–Veech group.

To formalise the idea of generating symplectic transvections from a set of generators using the previous lemma, we introduce the following definitions:

Definition 2.2.8. For $V \subseteq H_1(M_\pi \setminus \Sigma_\pi)$, we consider the group $G_V \subseteq \mathrm{Sp}(\Omega_\pi, \mathbb{Z})$ generated by the symplectic transvections $\{T_v\}_{v \in V}$. We say that a set $X \subseteq H_1(M_\pi \setminus \Sigma_\pi)$ is Ω_π*-closed* if $X = -X$ and $\langle v, w \rangle = 1$ implies $v + w, v - w \in X$ for every $v, w \in X$. We denote by V^{Ω_π} the smallest Ω_π-closed set containing V. By Lemma 2.2.6, $\{T_v\}_{v \in V^{\Omega_\pi}} \subseteq G_V$.

We also need a way to generate squares of symplectic transvections along sum of vectors when their intersection form is zero. We have the following:

Lemma 2.2.9. *Let* $v_1, v_2, v_3, v_4 \in H_1(M_\pi \setminus \Sigma_\pi)$ *such that*

$$\langle v_i, v_j \rangle_{i,j=1}^4 = \begin{pmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 1 \\ 0 & -1 & 0 & 1 \\ -1 & -1 & -1 & 0 \end{pmatrix}$$

then $T^2_{v_1+v_2}, T^2_{v_1-v_2}, T^2_{v_1+v_3}, T^2_{v_1-v_3} \in G_{\{v_1,v_2,v_3,v_4\}}$.

Proof. We will first show that $2v_1 + v_2, 2v_1 - v_3 \in \{v_1, v_2, v_3, v_4\}^{\Omega_\pi}$. For this, it is enough to observe that:

$$\begin{aligned} 1 = \langle v_1, v_4 \rangle &= \langle v_1 + v_4, -v_1 \rangle = \langle 2v_1 + v_4, -v_3 \rangle = \langle 2v_1 - v_3 + v_4, v_4 \rangle \\ &= \langle 2v_1 - v_3, v_2 \rangle = \langle 2v_1 + v_2 - v_3, v_3 \rangle. \end{aligned}$$

We obtain that $T_{2v_1+v_2}, T_{2v_1-v_3} \in G_{\{v_1,v_2,v_3,v_4\}}$. Now, we have that:

$$T_{2v_1+v_2}(u) = u + 4\langle v_1, u \rangle v_1 + 2\langle v_1, u \rangle v_2 + 2\langle v_2, u \rangle v_1 + \langle v_2, u \rangle v_2$$

and, since $\langle v_1, v_2 \rangle = \langle v_2, v_1 \rangle = 0$, we also have that:

$$T_{v_1}^{-2} T_{v_2} T_{2v_1+v_2}(u) = u + 2\langle v_1, u \rangle v_1 + 2\langle v_1, u \rangle v_2 + 2\langle v_2, u \rangle v_1 + 2\langle v_2, u \rangle v_2 = T^2_{v_1+v_2}(u).$$

Similarly, we have that:

$$T_{2v_1-v_3}(u) = u + 4\langle v_1, u \rangle v_1 - 2\langle v_1, u \rangle v_3 - 2\langle v_3, u \rangle v_1 + \langle v_3, u \rangle v_3$$

and that:

$$T_{v_1}^{-2}T_{v_3}T_{2v_1-v_3}(u) = u + 2\langle v_1, u\rangle v_1 - 2\langle v_1, u\rangle v_3 - 2\langle v_3, u\rangle v_1 + 2\langle v_3, u\rangle v_3 = T_{v_1-v_3}^2(u).$$

We can prove that $T_{v_1-v_2}^2, T_{v_1+v_3}^2 \in G_{\{v_1,v_2,v_3,v_4\}}$ in a similar way by swapping the roles of v_2 and v_3. □

The signs of the intersections are not important as shown by the following useful corollary:

Corollary 2.2.10. *Let $v_1', v_2', v_3', v_4' \in H_1(M_\pi \setminus \Sigma_\pi)$ such that*

$$|\langle v_i', v_j'\rangle|_{i,j=1}^4 = \begin{pmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 1 \\ 0 & 1 & 0 & 1 \\ 1 & 1 & 1 & 0 \end{pmatrix}$$

then $T_{v_1'+v_2'}^2, T_{v_1'-v_2'}^2, T_{v_1'+v_3'}^2, T_{v_1'-v_3'}^2 \in G_{\{v_1',v_2',v_3',v_4'\}}$.

Proof. We apply the previous lemma with $v_1 = \pm v_1'$, $\{v_2, v_3\} = \{\pm v_2', \pm v_3'\}$ and $v_4 = v_4'$, where we first choose the signs of v_1, v_2 and v_3 so $\langle v_1, v_4\rangle = \langle v_2, v_4\rangle = \langle v_3, v_4\rangle = 1$ and then we choose the order of v_2 and v_3 so $\langle v_2, v_3\rangle = 1$. □

Finally, we will prove that the Rauzy–Veech groups preserve the quadratic form Q_π defining the spin parity. By definition, $Q_\pi(e_\alpha) = 1$ for each $\alpha \in \mathcal{A}$, so given $u = \sum_{\alpha\in\mathcal{A}} u_\alpha e_\alpha$ we have that:

$$Q_\pi(u) = \sum_{\alpha<\beta} u_\alpha(\Omega_\pi)_{\alpha\beta}u_\beta + \sum_{\alpha\in\mathcal{A}} u_\alpha \bmod 2,$$

where the notation "$\sum_{\alpha<\beta}$" means "$\sum_{\pi_t(\alpha)<\pi_t(\beta)}$". By definition, this quadratic form satisfies $Q_\pi(u + v) = Q_\pi(u) + Q_\pi(v) + \langle u, v\rangle \bmod 2$.

Remark 2.2.11. The usual definition $Q_\pi(u) = \operatorname{ind}(c)+1 \bmod 2$, where c is a curve whose modulo-two homology class is u, only works when the order of every marked point is even. Nevertheless, we will always consider the explicit formula in terms of the basis $\{e_\alpha\}_{\alpha\in\mathcal{A}}$ as the definition of Q_π, which is valid in any case.

Throughout the entire chapter, we denote the modulo-two reduction of an object $\bullet$ by a bar above it: $\overline{\bullet}$. We can now prove that the Kontsevich-Zorich matrices preserve these quadratic forms.

Lemma 2.2.12. *Let $\gamma = \pi \to \pi'$ be an arrow of some Rauzy diagram. Then, $Q_{\pi'}(u) = Q_\pi(u\overline{B}_\gamma)$ for every $u \in H_1(M_{\pi'} \setminus \Sigma_{\pi'}; \mathbb{Z}/2\mathbb{Z})$.*

Proof. Let α_w and α_l be the winner and loser, respectively, for the Rauzy induction. Then, $\overline{B}_\gamma = \overline{\mathrm{Id}} + \overline{E}_{\alpha_l\alpha_w}$. Thus,

$$Q_\pi(u\overline{B}_\gamma) = Q_\pi(u + u_{\alpha_l}\overline{e}_{\alpha_w}) = Q_\pi(u) + u_{\alpha_l} + u_{\alpha_l}\langle u, \overline{e}_{\alpha_w}\rangle_\pi,$$

where $\langle\cdot,\cdot\rangle_\pi$ denotes the bilinear form induced by Ω_π.

On the other hand, since $\Omega_{\pi'} = B_\gamma \Omega_\pi B_\gamma^\intercal$ an immediate computation yields:

$$(\Omega_{\pi'})_{\alpha\beta} = \begin{cases} \Omega_{\alpha\beta} & \alpha \neq \alpha_1, \beta \neq \alpha_1 \\ \Omega_{\alpha\beta} + \Omega_{\alpha_{\mathrm{w}}\beta} & \alpha = \alpha_1, \beta \neq \alpha_1 \\ \Omega_{\alpha\beta} + \Omega_{\alpha\alpha_{\mathrm{w}}} & \alpha \neq \alpha_1, \beta = \alpha_1 \\ 0 & \alpha = \alpha_1 = \beta \end{cases}$$

and, therefore,

$$\begin{aligned} Q_{\pi'}(u) &= \sum_{\alpha<\beta} u_\alpha (\overline{\Omega}_{\pi'})_{\alpha\beta} u_\beta + \sum_{\alpha\in\mathcal{A}} u_\alpha = \sum_{\alpha<\beta} u_\alpha (\overline{\Omega}_\pi)_{\alpha\beta} u_\beta + u_{\alpha_1} \sum_{\alpha\neq\alpha_1} u_\alpha (\overline{\Omega}_\pi)_{\alpha\alpha_{\mathrm{w}}} + \sum_{\alpha\in\mathcal{A}} u_\alpha \\ &= \sum_{\alpha<\beta} u_\alpha (\overline{\Omega}_\pi)_{\alpha\beta} u_\beta + u_{\alpha_1} + u_{\alpha_1} \sum_{\alpha\in\mathcal{A}} u_\alpha (\overline{\Omega}_\pi)_{\alpha\alpha_{\mathrm{w}}} + \sum_{\alpha\in\mathcal{A}} u_\alpha \\ &= Q_\pi(u) + u_{\alpha_1} + u_{\alpha_1} \langle u, \bar{e}_{\alpha_{\mathrm{w}}} \rangle_\pi = Q_\pi(u\overline{B}_\gamma) \end{aligned}$$

where we used that $\overline{\Omega}_{\alpha_1\alpha_{\mathrm{w}}} = 1$. □

We obtain the following straightforward corollary:

Corollary 2.2.13. *The action on $H_1(M_\pi \setminus \Sigma_\pi; \mathbb{Z}/2\mathbb{Z})$ of the Rauzy–Veech group preserves Q_π. In other words, $\overline{\mathrm{RV}}(\pi) \subseteq \mathrm{O}(Q_\pi)$, where $\mathrm{O}(Q_\pi) \subseteq \mathrm{Sp}(\overline{\Omega}_\pi, \mathbb{Z}/2\mathbb{Z})$ is the orthogonal group induced by Q_π.*

We will show that, for some connected components of strata, the previous corollary characterises the Rauzy–Veech group. That is, the group is the preimage of $\mathrm{O}(Q_\pi)$ for the modulo-two reduction. Nevertheless, this is not true for every connected component. In particular, for $g \geq 3$ and every hyperelliptic connected component of $\mathcal{H}(2g-2)$ or $\mathcal{H}(g-1, g-1)$ there exist orthogonal transvections not belonging to the Rauzy–Veech group (see Section 2.4).

2.3 Generating the level-two congruence subgroup

The first part of our proof that the Rauzy–Veech groups of the even and odd connected components of minimal strata are the entire preimages of $\mathrm{O}(Q)$ by the modulo-two reduction consists of showing that $\ker(\mathrm{Sp}(\Omega_\pi, \mathbb{Z}) \to \mathrm{Sp}(\overline{\Omega}_\pi, \mathbb{Z}/2\mathbb{Z})) \subseteq \mathrm{RV}(\pi)$, where π represents a nonhyperelliptic connected component of $\mathcal{H}(2g-2)$. That is, we have to prove that the level-two congruence subgroup is contained in $\mathrm{RV}(\pi)$.

For this section, we will use the following explicit permutation representatives of minimal strata computed by Zorich [Zor08, Proposition 3, Proposition 4]:

$$\tau^{(g)} = \begin{pmatrix} 0 & 1 & 2 & 3 & 5 & 6 & \cdots & 3g-7 & 3g-6 & 3g-4 & 3g-3 \\ 3 & 2 & 6 & 5 & 9 & 8 & \cdots & 3g-3 & 3g-4 & 1 & 0 \end{pmatrix}$$

for $g \geq 3$ and

$$\sigma^{(g)} = \begin{pmatrix} 0 & 1 & 2 & 3 & 5 & 6 & \cdots & 3g-7 & 3g-6 & 3g-4 & 3g-3 \\ 6 & 5 & 3 & 2 & 9 & 8 & \cdots & 3g-3 & 3g-4 & 1 & 0 \end{pmatrix}$$

for $g \geq 4$. One has that $M_{\tau^{(g)}} \in \mathcal{H}(2g-2)^{\text{odd}}$ and that $M_{\sigma^{(g)}} \in \mathcal{H}(2g-2)^{\text{even}}$.

A finite set of generators of the level-two congruence subgroup is the following: given a symplectic basis $(b_\alpha)_{\alpha\in\mathcal{A}}$, the squares of symplectic transvections $T^2_{b_\alpha}$ and $T^2_{b_\alpha+b_\beta}$ for every $\alpha, \beta \in \mathcal{A}$ [Mum07, Appendix to Section 5]. We will prove this fact for the family $(\tau^{(g)})_{g\geq 3}$ and $(\sigma^{(g)})_{g\geq 4}$ separately to obtain that the associated Rauzy–Veech group contains the level-two congruence subgroup. Observe that the condition is redundant if $\alpha = \beta$, since $T^2_{2b_\alpha} = (T^2_{b_\alpha})^4$, so we can ignore these cases.

2.3.1 Proof for the first family

Our objective is now proving that the level-two congruence subgroup can be generated for the family $(\tau^{(g)})_{g\geq 3}$ of irreducible permutations. We will denote $M^{(g)}$ for $M_{\tau^{(g)}}$, $\Sigma^{(g)}$ for $\Sigma_{\tau^{(g)}}$ and $\Omega^{(g)}$ for $\Omega_{\tau^{(g)}}$ until explicitly stated otherwise.

We start by finding an appropriate symplectic basis:

Lemma 2.3.1. *If $g \geq 3$, a symplectic basis for $(H_1(M^{(g)} \setminus \Sigma^{(g)}), \langle\cdot,\cdot\rangle)$ is given by the following pairs of vectors:*

$$\begin{gathered} \{e_2, e_3\}, \{e_5, e_6\}, \{e_8, e_9\}, \dots, \{e_{3g-4}, e_{3g-3}\}, \\ \{e_0 + (-e_2 + e_3) + (-e_5 + e_6) + \cdots + (-e_{3g-4} + e_{3g-3}), \\ e_1 + (-e_2 + e_3) + (-e_5 + e_6) + \cdots + (-e_{3g-4} + e_{3g-3})\}. \end{gathered}$$

Proof. The proof is a straightforward computation. □

We denote the last two vectors in the previous lemma by v^* and w^*, respectively. The next series of lemmas show that $\mathrm{RV}(\tau^{(g)})$ contains the desired squares of symplectic transvections for this basis. Recall that if $v \in H_1(M^{(g)} \setminus \Sigma^{(g)})$ belongs to $\{e_\alpha\}^{\Omega^{(g)}}_{\alpha\in\mathcal{A}}$, then $T_v \in \mathrm{RV}(\tau^{(d)})$ (and, therefore, $T_v^2 \in \mathrm{RV}(\tau^{(g)})$) by Lemma 2.2.4 and Lemma 2.2.6. For this reason, we will sometimes prove that $v \in \{e_\alpha\}^{\Omega^{(g)}}_{\alpha\in\mathcal{A}}$. Moreover, if two vectors v, w belong to $\{e_\alpha\}^{\Omega^{(g)}}_{\alpha\in\mathcal{A}}$ and satisfy $\langle v, w\rangle = 1$, then $v + w$ belongs to $\{e_\alpha\}^{\Omega^{(g)}}_{\alpha\in\mathcal{A}}$, so in all of these cases we get that $T^2_{v+w} \in \mathrm{RV}(\tau^{(d)})$ "for free". This is the case for the vectors $e_2, e_3, e_5, e_6, \dots, e_{3g-4}, e_{3g-3}$ and also for $e_2 + e_3, e_5 + e_6, \dots, e_{3g-4} + e_{3g-3}$.

The following lemma is not a strict part of the proof, but will allow us to simplify many arguments.

Lemma 2.3.2. *If $\beta \bmod 6 = 3$, then $(-e_2 + e_3) + (-e_5 + e_6) + \cdots + (-e_{\beta-1} + e_\beta) \in \{e_\alpha\}^{\Omega^{(g)}}_{\alpha\in\mathcal{A}}$.*

Proof. Observe that $v_0 = -e_2 + e_3 \in \{e_\alpha\}_{\alpha\in\mathcal{A}}^{\Omega^{(g)}}$. We continue inductively: for $k \geq 1$ we have that:

$$\begin{aligned}
1 &= \langle e_0, e_{6k+3}\rangle = \langle -e_0 + e_{6k+3}, -e_{6k+5}\rangle = \langle -e_0 + e_{6k+3} - e_{6k+5}, -e_{6k+8}\rangle \\
&= \langle -e_0 + e_{6k+3} - e_{6k+5} - e_{6k+8}, e_0\rangle = \langle e_{6k+3} - e_{6k+5} - e_{6k+8}, v_k\rangle \\
&= \langle v_k + e_{6k+3} - e_{6k+5} - e_{6k+8}, -e_{6k+3}\rangle = \langle v_k - e_{6k+5} - e_{6k+8}, -e_{6k+6}\rangle \\
&= \langle v_k - e_{6k+5} + e_{6k+6} - e_{6k+8}, -e_{6k+9}\rangle
\end{aligned}$$

so $v_{k+1} = v_k + (-e_{6k+5} + e_{6k+6}) + (-e_{6k+8} + e_{6k+9}) \in \{e_\alpha\}_{\alpha\in\mathcal{A}}^{\Omega^{(g)}}$. By continuing this way until $k = (\beta - 3)/6$ we obtain the desired result. □

Lemma 2.3.3. *Let $\alpha, \beta \in \{2, 3, 5, 6, \ldots, 3g-4, 3g-3\}$ such that $\langle e_\alpha, e_\beta\rangle = 0$. Then, we have that $T^2_{e_\alpha+e_\beta} \in \mathrm{RV}(\tau^{(g)})$.*

Proof. Let $\beta' \in \mathcal{A} \setminus \{0, 1\}$ such that $|\langle e_\beta, e_{\beta'}\rangle| = 1$. We can use Corollary 2.2.10 with $v'_1 = e_\alpha$, $v'_2 = e_\beta$, $v'_3 = e_{\beta'}$, $v'_4 = e_0$. □

Lemma 2.3.4. *If $3g - 3 \bmod 6 = 3$, then $T^2_{v^*}, T^2_{w^*} \in \mathrm{RV}(\tau^{(g)})$ and $v^* + w^* \in \{e_\alpha\}_{\alpha\in\mathcal{A}}^{\Omega^{(g)}}$.*

If $3g - 3 \bmod 6 = 0$, then $v^*, w^*, v^* + w^* \in \{e_\alpha\}_{\alpha\in\mathcal{A}}^{\Omega^{(g)}}$.

Proof. If $3g-3 \bmod 6 = 3$, then $v = (-e_2+e_3)+\cdots+(-e_{3g-4}+e_{3g-3}) \in \{e_\alpha\}_{\alpha\in\mathcal{A}}^{\Omega^{(g)}}$ by Lemma 2.3.2. We can use Corollary 2.2.10 with $v'_1 = v$, $v'_2 = e_0$, $v'_3 = e_1$ and $v'_4 = e_2$ to conclude that $T^2_{v^*}, T^2_{w^*} \in \mathrm{RV}(\tau^{(g)})$. Moreover, we have that $v^* + w^* \in \{e_\alpha\}_{\alpha\in\mathcal{A}}^{\Omega^{(g)}}$ since

$$\begin{aligned}
1 &= \langle e_0, e_{3g-3}\rangle = \langle e_0 - e_{3g-3}, -v\rangle = \langle e_0 - e_{3g-3} + v, -v\rangle \\
&= \langle e_0 - e_{3g-3} + 2v, -e_{3g-3}\rangle = \langle e_0 + 2v, e_1\rangle
\end{aligned}$$

so $v^* + w^* = e_0 + e_1 + 2v \in \{e_\alpha\}_{\alpha\in\mathcal{A}}^{\Omega^{(g)}}$.

If $3g - 3 \bmod 6 = 0$, we have that $v = (-e_2 + e_3) + \cdots + (-e_{3g-7} + e_{3g-6}) \in \{e_\alpha\}_{\alpha\in\mathcal{A}}^{\Omega^{(g)}}$ by Lemma 2.3.2. Observe that:

$$\begin{aligned}
1 &= \langle e_1, e_{3g-6}\rangle = \langle e_1 - e_{3g-6}, e_{3g-4}\rangle = \langle e_1 - e_{3g-6} - e_{3g-4}, e_0\rangle \\
&= \langle e_0 + e_1 - e_{3g-6} - e_{3g-4}, -v\rangle = \langle e_0 + e_1 - e_{3g-6} - e_{3g-4} + v, e_{3g-6}\rangle \\
&= \langle e_0 + e_1 - e_{3g-4} + v, e_{3g-3}\rangle,
\end{aligned}$$

so $w = v + e_0 + e_1 + (-e_{3g-4} + e_{3g-3}) \in \{e_\alpha\}_{\alpha\in\mathcal{A}}^{\Omega^{(g)}}$. Since $\langle w, -e_0\rangle = 1$ and $\langle w, e_1\rangle = 1$, we obtain that $v^* = w - e_1$ and $w^* = w - e_0$ belong to $\{e_\alpha\}_{\alpha\in\mathcal{A}}^{\Omega^{(g)}}$. We also obtain that $v^* + w^* \in \{e_\alpha\}_{\alpha\in\mathcal{A}}^{\Omega^{(g)}}$ since $\langle v^*, w^*\rangle = 1$. □

The next lemma completes the proof:

Lemma 2.3.5. *Let $\beta \in \{2, 3, 5, 6, \ldots, 3g-4, 3g-3\}$. Then, $T^2_{v^*+e_\beta}, T^2_{w^*+e_\beta} \in \mathrm{RV}(\tau^{(g)})$.*

Proof. We consider two cases:

If $3g-3 \bmod 6 = 3$, we have that $v = (-e_2+e_3)+\cdots+(-e_{3g-4}+e_{3g-3}) \in \{e_\alpha\}_{\alpha\in\mathcal{A}}^{\Omega^{(g)}}$ by Lemma 2.3.2. Observe that $v^*+e_\beta \in \{e_\alpha\}_{\alpha\in\mathcal{A}}^{\Omega^{(g)}}$ since $1 = \langle e_0, e_\beta\rangle = \langle e_0+e_\beta, v\rangle$. We can prove that $w^*+e_\beta \in \{e_\alpha\}_{\alpha\in\mathcal{A}}^{\Omega^{(g)}}$ analogously.

If $3g-3 \bmod 6 = 0$, let $\beta' \in \mathcal{A}\setminus\{0,1\}$ such that $|\langle e_\beta, e_{\beta'}\rangle| = 1$. By the previous lemma, we have that $v^*, w^* \in \{e_\alpha\}_{\alpha\in\mathcal{A}}^{\Omega^{(g)}}$. We conclude by Corollary 2.2.10 with $v_1' = v^*$, $v_2' = e_\beta$, $v_3' = e_{\beta'}$ and $v_4' = e_1$ and with $v_1' = w^*$, $v_2' = e_\beta$, $v_3' = e_{\beta'}$ and $v_4' = e_0$. □

We obtain the following proposition:

Proposition 2.3.6. *For any $g \geq 3$, the symplectic transvections $\{T_\alpha\}_{\alpha\in\mathcal{A}}$ generate a group containing the level-two congruence subgroup* $\ker(\mathrm{Sp}(\Omega_{\tau^{(g)}},\mathbb{Z}) \to \mathrm{Sp}(\bar{\Omega}_{\tau^{(g)}},\mathbb{Z}/2\mathbb{Z}))$*. In particular, it is contained in* $\mathrm{RV}(\tau^{(g)})$*.*

2.3.2 Proof for the second family

We will now prove that the level-two congruence subgroup can be generated for the family $(\sigma^{(g)})_{g\geq 4}$ of irreducible permutations. The proof is very similar to the one for the first family, although some computations are different so it is necessary to present it in full detail. We will denote $M^{(g)}$ for $M_{\sigma^{(g)}}$, $\Sigma^{(g)}$ for $\Sigma_{\sigma^{(g)}}$ and $\Omega^{(g)}$ for $\Omega_{\sigma^{(g)}}$ until explicitly stated otherwise.

We start by finding an appropriate symplectic basis:

Lemma 2.3.7. *If $g \geq 4$, a symplectic basis for $(H_1(M^{(g)}\setminus\Sigma^{(g)}), \langle\cdot,\cdot\rangle)$ is given by the following pairs of vectors:*

$$\begin{gathered}\{e_2-e_5+e_6, e_3-e_5+e_6\}, \{e_5, e_6\}, \{e_8, e_9\}, \ldots, \{e_{3g-4}, e_{3g-3}\},\\ \{e_0+(-e_2+e_3)+(-e_5+e_6)+\cdots+(-e_{3g-4}+e_{3g-3}),\\ e_1+(-e_2+e_3)+(-e_5+e_6)+\cdots+(-e_{3g-4}+e_{3g-3})\}.\end{gathered}$$

Proof. The proof is a straightforward computation. □

As before, we denote the last two vectors by v^* and w^* respectively. Observe that we have that $e_5, e_6, \ldots, e_{3g-4}, e_{3g-3}$ and $e_5+e_6, \ldots, e_{3g-4}+e_{3g-3}$ belong to $\{e_\alpha\}_{\alpha\in\mathcal{A}}^{\Omega^{(g)}}$ "for free".

We have a lemma analogous to Lemma 2.3.2:

Lemma 2.3.8. *If $\beta \bmod 6 = 3$, then* $(-e_2+e_3)+(-e_5+e_6)+\cdots+(-e_{\beta-1}+e_\beta) \in \{e_\alpha\}_{\alpha\in\mathcal{A}}^{\Omega^{(g)}}$.

Proof. We will prove that $(-e_2+e_3)+(-e_5+e_6)+(-e_8+e_9) \in \{e_\alpha\}_{\alpha\in\mathcal{A}}^{\Omega^{(g)}}$. Then, we can continue inductively as in the proof of Lemma 2.3.2. We have that:

$$\begin{aligned}1 &= \langle e_0, e_3\rangle = \langle e_0+e_3, e_8\rangle = \langle e_0+e_3-e_8, e_1\rangle = \langle e_0-e_1+e_3-e_8, -e_2\rangle\\ &= \langle e_0-e_1-e_2+e_3-e_8, -e_9\rangle = \langle e_0-e_1-e_2+e_3-e_8+e_9, e_0\rangle\\ &= \langle -e_1-e_2+e_3-e_8+e_9, -e_5\rangle = \langle -e_1-e_2+e_3-e_5-e_8+e_9, e_1\rangle\end{aligned}$$

$$= \langle -e_2 + e_3 - e_5 - e_8 + e_9, -e_6 \rangle,$$

which shows what we wanted. □

The proofs of some lemmas are very similar to the ones for the first family (replacing the usage of Lemma 2.3.2 by Lemma 2.3.8) and in these cases we will just refer to such proofs. Indeed, one has that, for any $\alpha \in \{0, 1, 8, 9, 11, 12, \dots, 3g-3\}$, $\langle v, e_\alpha \rangle_{\tau^{(g)}} = \langle v, e_\alpha \rangle_{\sigma^{(g)}}$ for any v, so $\tau^{(g)}$ and $\sigma^{(g)}$ can only differ for $\alpha \in \{2, 3, 5, 6\}$ (and sometimes they are equal even in this case). We will implicitly exploit this fact to avoid having to repeat the proofs.

Lemma 2.3.9. *Let $\alpha, \beta \in \{5, 6, \dots, 3g-4, 3g-3\}$ such that $\langle e_\alpha, e_\beta \rangle = 0$. Then, we have that $T^2_{e_\alpha + e_\beta} \in \mathrm{RV}(\sigma^{(g)})$.*

Proof. The proof is identical to that of Lemma 2.3.3. □

Lemma 2.3.10. *If $3g-3 \bmod 6 = 3$, then $T^2_{v^*}, T^2_{w^*} \in \mathrm{RV}(\sigma^{(g)})$ and $v^* + w^* \in \{e_\alpha\}^{\Omega^{(g)}}_{\alpha \in \mathscr{A}}$.*
If $3g-3 \bmod 6 = 0$, then $v^*, w^*, v^* + w^* \in \{e_\alpha\}^{\Omega^{(g)}}_{\alpha \in \mathscr{A}}$.

Proof. The proof is identical to that of Lemma 2.3.4. □

Lemma 2.3.11. *Let $\beta \in \{5, 6, \dots, 3g-4, 3g-3\}$. Then, $T^2_{v^*+e_\beta}, T^2_{w^*+e_\beta} \in \mathrm{RV}(\sigma^{(g)})$.*

Proof. The proof is identical to that of Lemma 2.3.5. □

Lemma 2.3.12. *We have that $T^2_{e_\beta - e_5 + e_6} \in \mathrm{RV}(\sigma^{(g)})$ and $(e_2 - e_5 + e_6) + (e_3 - e_5 + e_6) \in \{e_\alpha\}^{\Omega^{(g)}}_{\alpha \in \mathscr{A}}$ for any $\beta \in \{2, 3\}$.*

Proof. Observe that $T^2_{e_2 - e_5 + e_6}, T^2_{e_3 - e_5 + e_6} \in \mathrm{RV}(\sigma^{(g)})$ by Corollary 2.2.10 with $v'_1 = e_5 - e_6$, $v'_2 = e_2$, $v'_3 = e_3$ and $v'_4 = e_5$. Moreover, we have that:

$$1 = \langle e_2, e_5 \rangle = \langle e_2 - e_5, e_5 \rangle = \langle e_2 - 2e_5, -e_6 \rangle = \langle e_2 - 2e_5 + e_6, -e_6 \rangle = \langle e_2 - 2e_5 + 2e_6, e_3 \rangle,$$

so $(e_2 - e_5 + e_6) + (e_3 - e_5 + e_6) \in \{e_\alpha\}^{\Omega^{(g)}}_{\alpha \in \mathscr{A}}$. □

The following lemma completes the proof:

Lemma 2.3.13. *We have that $(e_\beta - e_5 + e_6) + e_{\alpha'} \in \{e_\alpha\}^{\Omega^{(g)}}_{\alpha \in \mathscr{A}}$ for any letters $\beta \in \{2, 3\}$ and $\alpha' \in \{5, 6, \dots, 3g-4, 3g-3\}$. Moreover, $T^2_{(e_\beta - e_5 + e_6)+v^*}, T^2_{(e_\beta - e_5 + e_6)+w^*} \in \mathrm{RV}(\sigma^{(g)})$.*

Proof. If $\alpha' = 5$, the result is obvious. If $\alpha' = 6$, it is enough to observe that

$$1 = \langle e_\beta, e_6 \rangle = \langle e_\beta + e_6, e_6 \rangle = \langle e_\beta + 2e_6, -e_5 \rangle.$$

If $\alpha' \in \{8, 9, \dots, 3g-4, 3g-3\}$, let $\beta' \in \mathscr{A} \setminus \{0, 1\}$ such that $\varepsilon = \langle e_{\alpha'}, e_{\beta'} \rangle \neq 0$. If $\varepsilon = 1$ we can use that:

$$1 = \langle e_1, e_\beta \rangle = \langle -e_1 + e_\beta, -e_{\alpha'} \rangle = \langle -e_1 + e_\beta + e_{\alpha'}, -e_0 \rangle = \langle -e_0 - e_1 + e_\beta + e_{\alpha'}, -e_5 \rangle$$

$$= \langle -e_0 - e_1 + e_\beta - e_5 + e_{\alpha'}, -e_{\beta'} \rangle = \langle -e_0 - e_1 + e_\beta - e_5 + e_{\alpha'} - e_{\beta'}, e_0 \rangle$$
$$= \langle -e_1 + e_\beta - e_5 + e_{\alpha'} - e_{\beta'}, -e_6 \rangle = \langle -e_1 + e_\beta - e_5 + e_6 + e_{\alpha'} - e_{\beta'}, -e_1 \rangle$$
$$= \langle e_\beta - e_5 + e_6 + e_{\alpha'} - e_{\beta'}, e_{\beta'} \rangle$$

to show that $(e_\beta - e_5 + e_6) + e_{\alpha'} \in \{e_\alpha\}^{\Omega^{(g)}}_{\alpha \in \mathcal{A}}$. The case $\varepsilon = -1$ can be proved similarly.

If $3g - 3 \bmod 6 = 3$, let $v = (-e_2 + e_3) + (-e_5 + e_6) + \cdots + (e_{3g-4} - e_{3g-3}) \in \{e_\alpha\}^{\Omega^{(g)}}_{\alpha \in \mathcal{A}}$ by Lemma 2.3.8. We have that $1 = \langle e_\beta, v \rangle = \langle e_\beta + v, -e_0 \rangle$, so $v^* + e_\beta \in \{e_\alpha\}^{\Omega^{(g)}}_{\alpha \in \mathcal{A}}$. We can use Corollary 2.2.10 with $v_1' = -e_5 + e_6$, $v_2' = v^* + e_2$, $v_3' = v^* + e_3$ and $v_4' = e_1 + e_5$ to conclude that $T^2_{(e_\beta - e_5 + e_6) + v^*} \in \mathrm{RV}(\sigma^{(g)})$. The fact that $T^2_{(e_\beta - e_5 + e_6) + w^*} \in \mathrm{RV}(\sigma^{(g)})$ can be proved similarly.

Finally, if $3g - 3 \bmod 6 = 0$ we have that $v^*, w^* \in \{e_\alpha\}^{\Omega^{(g)}}_{\alpha \in \mathcal{A}}$. Let $v \in \{v^*, w^*\}$. We will show that $(e_\beta - e_5 + e_6) + v \in \{e_\alpha\}^{\Omega^{(g)}}_{\alpha \in \mathcal{A}}$. If $v = v^*$, we put $\varepsilon = 1$, $w = e_1$ and $w' = e_0$. If $v = w^*$, we put $\varepsilon = -1$, $w = e_0$ and $w' = e_1$. Observe that $\langle v, w \rangle = \varepsilon$, that $\langle v, w' \rangle = 0$ and that:

$$1 = \langle e_0, e_\beta \rangle = \langle -e_0 + e_\beta, -e_{3g-3} \rangle = \langle -e_0 + e_\beta - e_{3g-3}, -e_1 \rangle$$
$$= \langle -e_0 - e_1 + e_\beta - e_{3g-3}, -e_5 \rangle = \langle -e_0 - e_1 + e_\beta - e_5 - e_{3g-3}, \varepsilon v \rangle$$
$$= \langle -e_0 - e_1 + e_\beta - e_5 - e_{3g-3} + v, w \rangle = \langle -e_0 - e_1 + e_\beta - e_5 - e_{3g-3} + v + w, -e_6 \rangle$$
$$= \langle -e_0 - e_1 + e_\beta - e_5 + e_6 - e_{3g-3} + v + w, -e_{3g-3} \rangle$$
$$= \langle -e_0 - e_1 + e_\beta - e_5 + e_6 + v + w, -w' \rangle.$$

Since $w + w' - e_0 - e_1 = 0$ in any case, we obtain that $e_\beta - e_5 + e_6 + v \in \{e_\alpha\}^{\Omega^{(g)}}_{\alpha \in \mathcal{A}}$. □

We obtain the following proposition:

Proposition 2.3.14. *For any $g \geq 4$, the symplectic transvections $\{T_\alpha\}_{\alpha \in \mathcal{A}}$ generate a group containing the level-two congruence subgroup $\ker(\mathrm{Sp}(\Omega_{\sigma^{(g)}}, \mathbb{Z}) \to \mathrm{Sp}(\overline{\Omega}_{\sigma^{(g)}}, \mathbb{Z}/2\mathbb{Z}))$. In particular, it is contained in $\mathrm{RV}(\sigma^{(g)})$.*

2.4 Generating the orthogonal group

For this section and the next, we will use some other permutations representing the even and odd components of minimal strata. Indeed, we consider the following two families of irreducible permutations:

$$\tau^{(d)} = \begin{pmatrix} 1 & 2 & 3 & \cdots & d-5 & d-4 & d-3 & d-2 & d-1 & d \\ d & d-1 & d-2 & \cdots & 6 & 3 & 2 & 5 & 4 & 1 \end{pmatrix}$$

for $d \geq 6$ and

$$\sigma^{(d)} = \begin{pmatrix} 1 & 2 & 3 & \cdots & d-7 & d-6 & d-5 & d-4 & d-3 & d-2 & d-1 & d \\ d & d-1 & d-2 & \cdots & 8 & 3 & 2 & 7 & 6 & 5 & 4 & 1 \end{pmatrix}$$

for $d \geq 8$.

These two families of permutations are similar in many ways. For the sake of simplicity and brevity, for this section and the next we write $\pi^{(d)}$ to refer to either $\tau^{(d)}$ or $\sigma^{(d)}$ and we set $M^{(d)} = M_{\pi^{(d)}}$, $\Sigma^{(d)} = \Sigma_{\pi^{(d)}}$, $\Omega^{(d)} = \Omega_{\pi^{(d)}}$ and $Q^{(d)} = Q_{\pi^{(d)}}$. Moreover, we will say that a vector $u \in H_1(M^{(d)} \setminus \Sigma^{(d)}; \mathbb{Z}/2\mathbb{Z})$ is *singular* if $Q^{(d)}(u) = 0$ and *nonsingular* otherwise.

We have that the genus g_d of $M^{(d)}$ is $d/2$ if d is even and $(d-1)/2$ if d is odd. In the latter case, the kernel of the symplectic form is generated by $e_\sharp = (1, -1, 1, \ldots, -1, 1)$.

Moreover, using the notation from the classification of connected components [KZ03], we have that:

Lemma 2.4.1. *We have that:*

$$M_{\tau^{(d)}} \in \begin{cases} \mathcal{H}(2g_d-2)^{\text{odd}} & d \bmod 8 \in \{0,6\} \\ \mathcal{H}(g_d-1, g_d-1)^{\text{nonhyp}} & d \bmod 8 \in \{1,5\} \\ \mathcal{H}(2g_d-2)^{\text{even}} & d \bmod 8 \in \{2,4\} \\ \mathcal{H}(g_d-1, g_d-1)^{\text{even}} & d \bmod 8 = 3 \\ \mathcal{H}(g_d-1, g_d-1)^{\text{odd}} & d \bmod 8 = 7 \end{cases} \qquad \textit{for every } d \geq 6 \textit{ and}$$

$$M_{\sigma^{(d)}} \in \begin{cases} \mathcal{H}(2g_d-2)^{\text{even}} & d \bmod 8 \in \{0,6\} \\ \mathcal{H}(g_d-1, g_d-1)^{\text{nonhyp}} & d \bmod 8 \in \{1,5\} \\ \mathcal{H}(2g_d-2)^{\text{odd}} & d \bmod 8 \in \{2,4\} \\ \mathcal{H}(g_d-1, g_d-1)^{\text{odd}} & d \bmod 8 = 3 \\ \mathcal{H}(g_d-1, g_d-1)^{\text{even}} & d \bmod 8 = 7 \end{cases} \qquad \textit{for every } d \geq 8.$$

Proof. Observe that if $d \bmod 8 \in \{1, 5\}$, then neither $M_{\tau^{(d)}}$ nor $M_{\sigma^{(d)}}$ are hyperelliptic, so they belong to the only nonhyperelliptic connected component of the stratum.

For the rest of the cases, we can compute the Arf invariant of $Q^{(d)}$ to determine the connected component in which $M^{(d)}$ lies. First, it is easy to see that if d is odd, then the Arf invariant of $Q^{(d)}$ is the same as that of $Q^{(d-1)}$, since any symplectic basis in $(d-1)$ dimensions is a maximal symplectic set in d dimensions. Therefore, we only need to compute the Arf invariant for even d.

We will use the definition of the Arf invariant as the value assumed most often by the quadratic form. We can count the number of nonsingular vectors by establishing an appropriate recurrence. To this end, we define the following quantities:

- the number $|\mathrm{NS}_0(Q^{(d)})|$ of nonsingular vectors with an even number of 1s;
- the number $|\mathrm{NS}_1(Q^{(d)})|$ of nonsingular vectors with an odd number of 1s;
- the number $|\mathrm{S}_0(Q^{(d)})|$ of singular vectors with an even number of 1s;
- the number $|\mathrm{S}_1(Q^{(d)})|$ of singular vectors with an odd number of 1s.

Then, we have the following recurrence:

$$|\mathrm{NS}_0(Q^{(d+1)})| = |\mathrm{NS}_0(Q^{(d)})| + |\mathrm{NS}_1(Q^{(d)})|, \quad |\mathrm{NS}_1(Q^{(d+1)})| = |\mathrm{NS}_1(Q^{(d)})| + |\mathrm{S}_0(Q^{(d)})|$$
$$|\mathrm{S}_0(Q^{(d+1)})| = |\mathrm{S}_0(Q^{(d)})| + |\mathrm{S}_1(Q^{(d)})|, \quad |\mathrm{S}_1(Q^{(d+1)})| = |\mathrm{S}_1(Q^{(d)})| + |\mathrm{NS}_0(Q^{(d)})|.$$

Indeed, given a nonsingular vector of $Q^{(d)}$ with an even number of 1s we construct a nonsingular vector of $Q^{(d+1)}$ with an even number of 1s by setting the last coordinate to be zero. Moreover, for any nonsingular vector of $Q^{(d)}$ with an odd number of 1s we construct a nonsingular vector of $Q^{(d+1)}$ with an even number of 1s by setting the last coordinate to be 1. This proves that $|\mathrm{NS}_0(Q^{(d+1)})| = |\mathrm{NS}_0(Q^{(d)})| + |\mathrm{NS}_1(Q^{(d)})|$, and the other relations can be shown similarly.

We first assume $\pi^{(d)} = \tau^{(d)}$. We have the following base cases for the recurrence (see Appendix A for some explicit computations): $|\mathrm{NS}_0(Q^{(6)})| = 16$, $|\mathrm{NS}_1(Q^{(6)})| = 20$, $|\mathrm{S}_0(Q^{(6)})| = 16$ and $|\mathrm{S}_1(Q^{(6)})| = 12$. The linear recurrence can be then solved, yielding that:

$$|\mathrm{NS}_0(Q^{(d)})| = 2^{d-2} + 2^{(d-2)/2}\cos\left(\frac{d\pi}{4}\right),\quad |\mathrm{NS}_1(Q^{(d)})| = 2^{d-2} - 2^{(d-2)/2}\sin\left(\frac{d\pi}{4}\right)$$
$$|\mathrm{S}_0(Q^{(d)})| = 2^{d-2} - 2^{(d-2)/2}\cos\left(\frac{d\pi}{4}\right),\quad |\mathrm{S}_1(Q^{(d)})| = 2^{d-2} + 2^{(d-2)/2}\sin\left(\frac{d\pi}{4}\right)$$

and therefore that the number of nonsingular vectors is:

$$|\mathrm{NS}(Q^{(d)})| = |\mathrm{NS}_0(Q^{(d+1)})| = 2^{d-1} + 2^{(d-1)/2}\cos\left(\frac{(d+1)\pi}{4}\right).$$

Recall that we are assuming that d is even, so $(d+1)$ is odd and $|\mathrm{NS}(Q^{(d)})| \neq 2^{d-1}$. By analysing the sign of $\cos((d+1)\pi/4)$, we obtain that there are more nonsingular than singular vectors if $d \bmod 8 \in \{0, 6\}$.

Now assume $\pi^{(d)} = \sigma^{(d)}$. We have the following base cases for the recurrence (see Appendix A for some explicit computations): $|\mathrm{NS}_0(Q^{(8)})| = 56$, $|\mathrm{NS}_1(Q^{(8)})| = 64$, $|\mathrm{S}_0(Q^{(8)})| = 72$ and $|\mathrm{S}_1(Q^{(8)})| = 64$. We conclude that:

$$|\mathrm{NS}_0(Q^{(d)})| = 2^{d-2} - 2^{(d-2)/2}\cos\left(\frac{d\pi}{4}\right),\quad |\mathrm{NS}_1(Q^{(d)})| = 2^{d-2} + 2^{(d-2)/2}\sin\left(\frac{d\pi}{4}\right)$$
$$|\mathrm{S}_0(Q^{(d)})| = 2^{d-2} + 2^{(d-2)/2}\cos\left(\frac{d\pi}{4}\right),\quad |\mathrm{S}_1(Q^{(d)})| = 2^{d-2} - 2^{(d-2)/2}\sin\left(\frac{d\pi}{4}\right),$$

so the number of nonsingular vectors is:

$$|\mathrm{NS}(Q^{(d)})| = |\mathrm{NS}_0(Q^{(d+1)})| = 2^{d-1} - 2^{(d-1)/2}\cos\left(\frac{(d+1)\pi}{4}\right).$$

A similar analysis as in the previous case shows that there are more nonsingular than singular vectors for if $d \bmod 8 \in \{2, 4\}$. □

Now, we need to prove that some specific elements of $H_1(M^{(d)} \setminus \Sigma^{(d)})$ belong to $\{e_\alpha\}^{\Omega}_{\alpha\in\mathcal{A}}$.

Lemma 2.4.2. *If* $\beta = 2 \bmod 4$, *then* $v = \sum_{\alpha=1}^{\beta}(-1)^\alpha e_\alpha \in \{e_\alpha\}^{\Omega^{(d)}}_{\alpha\in\mathcal{A}}$.

Proof. First assume that $\pi^{(d)} = \tau^{(d)}$. We have that:

$$1 = \langle e_2, -e_1\rangle = \langle -e_1 + e_2, -e_4\rangle = \langle -e_1 + e_2 + e_4, e_6\rangle = \langle -e_1 + e_2 + e_4 + e_6, -e_3\rangle,$$

so $v_0 = -e_1+e_2-e_3+e_4+e_6 \in \{e_\alpha\}_{\alpha\in\mathcal{A}}^{\Omega^{(d)}}$. If $\beta = 6$, we conclude since $v = v_0-e_5$ and $1 = \langle v_0, -e_5\rangle$. We continue inductively: if $\beta = 6 + 4k$ with $k \geq 1$, we have that

$$\begin{aligned}
1 &= \langle v_{k-1}, e_{3+4k}\rangle = \langle v_{k-1} - e_{3+4k}, -e_4\rangle = \langle v_{k-1} - e_{3+4k} - e_4, -e_{4+4k}\rangle \\
&= \langle v_{k-1} - e_{3+4k} - e_4 + e_{4+4k}, -e_3\rangle = \langle v_{k-1} - e_{3+4k} - e_4 + e_{4+4k} + e_3, e_{5+4k}\rangle \\
&= \langle v_{k-1} - e_{3+4k} - e_4 + e_{4+4k} + e_3 - e_{5+4k}, -e_4\rangle \\
&= \langle v_{k-1} - e_{3+4k} - e_4 + e_{4+4k} + e_3 - e_{5+4k} + e_4, e_{6+4k}\rangle \\
&= \langle v_{k-1} - e_{3+4k} - e_4 + e_{4+4k} + e_3 - e_{5+4k} + e_4 + e_{6+4k}, -e_3\rangle
\end{aligned}$$

and we obtain that $v_k = v_{k-1} - e_{3+4k} + e_{4+4k} - e_{5+4k} + e_{6+4k} \in \{e_\alpha\}_{\alpha\in\mathcal{A}}^{\Omega^{(d)}}$. Since $v = v_k - e_5$ and $\langle v_k, -e_5\rangle = 1$, we obtain the desired result.

Now assume that $\pi^{(d)} = \sigma^{(d)}$. Observe that:

$$\begin{aligned}
1 &= \langle e_2, -e_1\rangle = \langle -e_1 + e_2, -e_4\rangle = \langle -e_1 + e_2 + e_4, e_8\rangle = \langle -e_1 + e_2 + e_4 + e_8, -e_3\rangle \\
&= \langle -e_1 + e_2 - e_3 + e_4 + e_8, -e_5\rangle = \langle -e_1 + e_2 - e_3 + e_4 - e_5 + e_8, -e_8\rangle \\
&= \langle -e_1 + e_2 - e_3 + e_4 - e_5, -e_6\rangle,
\end{aligned}$$

so $-e_1 + e_2 + e_4 + e_8, \sum_{\alpha=1}^{6}(-1)^\alpha e_\alpha \in \{e_\alpha\}_{\alpha\in\mathcal{A}}^{\Omega^{(d)}}$. Now we will assume that $\beta \geq 10$.

We have that:

$$\begin{aligned}
1 &= \langle -e_1 + e_2 + e_4 + e_8, -e_5\rangle = \langle -e_1 + e_2 + e_4 - e_5 + e_8, e_9\rangle \\
&= \langle -e_1 + e_2 + e_4 - e_5 + e_8 - e_9, -e_6\rangle \\
&= \langle -e_1 + e_2 + e_4 - e_5 + e_6 + e_8 - e_9, e_{10}\rangle \\
&= \langle -e_1 + e_2 + e_4 - e_5 + e_6 + e_8 - e_9 + e_{10}, -e_3\rangle
\end{aligned}$$

so $v_0 = -e_1 + e_2 - e_3 + e_4 - e_5 + e_6 + e_8 - e_9 + e_{10} \in \{e_\alpha\}_{\alpha\in\mathcal{A}}^{\Omega^{(d)}}$. If $\beta = 10$, we conclude since $v = v_0 - e_7$ and $1 = \langle v_0, -e_7\rangle$. We continue inductively: if $\beta = 10 + 4k$ with $k \geq 1$, we have that

$$\begin{aligned}
1 &= \langle v_{k-1}, e_{7+4k}\rangle = \langle v_{k-1} - e_{7+4k}, -e_6\rangle = \langle v_{k-1} - e_{7+4k} - e_6, -e_{8+4k}\rangle \\
&= \langle v_{k-1} - e_{7+4k} - e_6 + e_{8+4k}, -e_3\rangle = \langle v_{k-1} - e_{7+4k} - e_6 + e_{8+4k} + e_3, e_{9+4k}\rangle \\
&= \langle v_{k-1} - e_{7+4k} - e_6 + e_{8+4k} + e_3 - e_{9+4k}, -e_6\rangle \\
&= \langle v_{k-1} - e_{7+4k} - e_6 + e_{8+4k} + e_3 - e_{9+4k} + e_6, e_{10+4k}\rangle \\
&= \langle v_{k-1} - e_{7+4k} - e_6 + e_{8+4k} + e_3 - e_{9+4k} + e_6 + e_{10+4k}, -e_3\rangle
\end{aligned}$$

and we obtain that $v_k = v_{k-1} - e_{7+4k} + e_{8+4k} - e_{9+4k} + e_{10+4k} \in \{e_\alpha\}_{\alpha\in\mathcal{A}}^{\Omega^{(d)}}$. Since $v = v_k - e_7$ and

$\langle v_k, -e_7\rangle = 1$, we obtain the desired result. □

For the rest of this section, we will fix d until explicitly stated otherwise and write M for $M^{(d)}$, Σ for $\Sigma^{(d)}$, etc. Recall that Q is the usual quadratic form on $H_1(M \setminus \Sigma; \mathbb{Z}/2\mathbb{Z})$ and that:

$$Q(u) = \sum_{\alpha<\beta} u_\alpha \overline{\Omega}_{\alpha\beta} u_\beta + \sum_{\alpha\in\mathcal{A}} u_\alpha.$$

We also denote by $\langle\cdot,\cdot\rangle$ the bilinear form on $H_1(M \setminus \Sigma; \mathbb{Z}/2\mathbb{Z})$ induced by $\overline{\Omega}$. Recall that Q satisfies $Q(u+v) = Q(u) + Q(v) + \langle u, v\rangle$ for every $u, v \in H_1(M \setminus \Sigma; \mathbb{Z}/2\mathbb{Z})$.

Definition 2.4.3. For $v \in H_1(M \setminus \Sigma; \mathbb{Z}/2\mathbb{Z})$ such that $Q(v) = 1$, we define the *orthogonal transvection* along v as:

$$\overline{T}_v(u) = u + \langle u, v\rangle v.$$

Observe that $\overline{T}_v^2 = \mathrm{Id}$ for each v since $\overline{T}_v^2(u) = \overline{T}_v(u + \langle u, v\rangle v) = u + \langle u, v\rangle v + \langle u, v\rangle v = u$. Moreover, it is not hard to see that if $v \in H_1(M \setminus \Sigma)$ then $\overline{T}_{\bar{v}}$ belongs to $O(Q)$ if and only if $Q(v) = 1$, so the set of orthogonal transvections coincides with the modulo-two reduction of the set of the symplectic transvections whose reductions preserve Q.

Lemma 2.4.4. *Let $v, w \in H_1(M_\pi \setminus \Sigma_\pi; \mathbb{Z}/2\mathbb{Z})$ such that $Q(v) = Q(w) = Q(v+w) = 1$. Then, $\overline{T}_v\overline{T}_w\overline{T}_v = \overline{T}_w\overline{T}_v\overline{T}_w = \overline{T}_{v+w}$.*

Proof. Since $Q(v+w) = Q(v) + Q(w) + \langle v, w\rangle$, we obtain that $\langle v, w\rangle = \langle w, v\rangle = 1$. The proof then follows from Lemma 2.2.6, since we can find $\tilde{v}, \tilde{w} \in H_1(M_\pi \setminus \Sigma_\pi)$ such that $\langle \tilde{v}, \tilde{w}\rangle = 1$ and $v = \tilde{v} \bmod 2$, $w = \tilde{w} \bmod 2$. □

We denote the set of nonsingular vectors by $\mathrm{NS}(Q) \subseteq H_1(M \setminus \Sigma; \mathbb{Z}/2\mathbb{Z})$ and the set of singular vectors by $\mathrm{S}(Q) \subseteq H_1(M \setminus \Sigma; \mathbb{Z}/2\mathbb{Z})$.

Definition 2.4.5. For $V \subseteq \mathrm{NS}(Q)$, let $\overline{G}_V \subseteq \mathrm{O}(Q)$ be the group spanned by the orthogonal transvections $\{\overline{T}_v\}_{v\in V}$. We say that a set $X \subseteq \mathrm{NS}(Q)$ is *Q-closed* if $Q(v+w) = 1$ implies $v+w \in X$ for every $v, w \in X$. We denote by V^Q the smallest Q-closed set containing V. By the previous lemma, $\{\overline{T}_v\}_{v\in V^Q} \subseteq \overline{G}_V$.

Observe that this definition is analogous to Definition 2.2.8. Indeed, the modulo-two reduction of a nonsingular Ω-closed set is a Q-closed set.

It is known from the theory of classical groups that orthogonal transvections generate $\mathrm{O}(Q)$ if the dimension of the vector space is at least 6 and the alternate form is nondegenerate [Gro02, Theorem 14.16]. This holds for every even d. Since $\overline{T}_\alpha = \overline{T}_{\bar{e}_\alpha}$ belongs to $\overline{\mathrm{RV}}(\pi)$ for every $\alpha \in \mathcal{A}$, it is enough to prove that $\{\bar{e}_\alpha\}^Q_{\alpha\in\mathcal{A}} = \mathrm{NS}(Q)$ to obtain that $\overline{\mathrm{RV}}(\pi) = \mathrm{O}(Q)$. We will need one extra auxiliary lemma:

Lemma 2.4.6. *If $d \bmod 4 = 2$, then $v = \sum_{\alpha\in\mathcal{A}} \bar{e}_\alpha$, $v_\beta = \sum_{\alpha\neq\beta} \bar{e}_\alpha \in \{\bar{e}_\alpha\}^Q_{\alpha\in\mathcal{A}}$ for each $\beta \in \mathcal{A}$.*

Proof. We have that $\sum_{\alpha\in\mathcal{A}}(-1)^\alpha e_\alpha \in \{e_\alpha\}^{\Omega}_{\alpha\in\mathcal{A}}$ by Lemma 2.4.2. By taking the modulo-two reduction, this implies that $v = \sum_{\alpha\in\mathcal{A}} \bar{e}_\alpha \in \{\bar{e}_\alpha\}^{Q}_{\alpha\in\mathcal{A}}$.

Now, for any $\beta \in \mathcal{A}$ we have that:

$$Q(v_\beta) = Q(v + e_\beta) = 1 + 1 + \sum_{\alpha\in\mathcal{A}} \langle e_\alpha, e_\beta\rangle = 1,$$

so $v_\beta \in \{\bar{e}_\alpha\}^{Q}_{\alpha\in\mathcal{A}}$. □

Observe that $\sum_{\alpha\in\mathcal{A}} \bar{e}_\alpha \in \mathrm{NS}(Q^{(d)})$ if and only if $d \bmod 4 \in \{1, 2\}$. We can now prove the final lemma of this section:

Lemma 2.4.7. *For each d, we have $\{\bar{e}_\alpha\}^{Q}_{\alpha\in\mathcal{A}} = \mathrm{NS}(Q) \setminus \ker\bar{\Omega}$.*

Proof. The proof is by induction. The base cases, $d = 6$ for $\tau^{(d)}$ and $d = 8$ for $\sigma^{(d)}$, can be done computationally (see Appendix A for computations). The inductive step is exactly the same for both families of irreducible permutations.

We have that

$$\bar{\Omega}^{(d)} = \begin{pmatrix} \bar{\Omega}^{(d-1)} & \bar{e}_\sharp^{\intercal} \\ \bar{e}_\sharp & 0 \end{pmatrix}$$

where $\bar{e}_\sharp = \sum_{\alpha<d} \bar{e}_\alpha$. Observe that, for any $u \in H_1(M^{(d)} \setminus \Sigma^{(d)}; \mathbb{Z}/2\mathbb{Z})$,

$$Q^{(d)}(u) = \sum_{\alpha<\beta<d} u_\alpha \bar{\Omega}^{(d-1)}_{\alpha\beta} u_\beta + u_d \sum_{\alpha<d} u_\alpha + \sum_{\alpha\in\mathcal{A}} u_\alpha = \sum_{\alpha<\beta<d} u_\alpha \bar{\Omega}^{(d-1)}_{\alpha\beta} u_\beta + (1 + u_d)\sum_{\alpha<d} u_\alpha + u_d.$$

Let $v \in H_1(M^{(d)} \setminus \Sigma^{(d)}; \mathbb{Z}/2\mathbb{Z})$ be such that $v_\alpha = u_\alpha$ for $\alpha < d$ and $v_d = 0$. We define the vector $p_{d-1}(u) \in H_1(M^{(d-1)} \setminus \Sigma^{(d-1)}; \mathbb{Z}/2\mathbb{Z})$ to be the projection of u by removing the d-th coordinate. Then:

$$Q^{(d)}(v) = \sum_{\alpha<\beta<d} u_\alpha \bar{\Omega}^{(d-1)}_{\alpha\beta} u_\beta + \sum_{\alpha<d} u_\alpha = Q^{(d-1)}(p_{d-1}(u)), \text{ so } Q^{(d)}(u) = Q^{(d)}(v) + u_d \sum_{\alpha\in\mathcal{A}} u_\alpha.$$

Assume now that $u \in \mathrm{NS}(Q^{(d)}) \setminus \ker\bar{\Omega}$. We consider several cases:

If $u_d = 0$, we have that $Q^{(d-1)}(p_{d-1}(u)) = 1$. If d is odd, then $\bar{\Omega}^{(d-1)}$ is invertible, so we have that $p_{d-1}(u) \notin \ker\bar{\Omega}^{(d-1)}$ and we obtain that $u \in \{\bar{e}_\alpha\}^{Q^{(d)}}_{\alpha\in\mathcal{A}}$. If, on the contrary, d is even, then $\ker\bar{\Omega}^{(d-1)} = \{0, \bar{e}_\sharp\}$. If $u = (\bar{e}_\sharp\ \ 0)$, then $d = 2 \bmod 4$ and can use the previous lemma to see that $u \in \{\bar{e}_\alpha\}^{Q^{(d)}}_{\alpha\in\mathcal{A}}$. If $u \neq (\bar{e}_\sharp\ \ 0)$, then $p_{d-1}(u) \notin \ker\bar{\Omega}^{(d-1)}$, so we can use the induction hypothesis.

If $u_d = 1$, we have that $u = v + \bar{e}_d$. If $\sum_{\alpha\in\mathcal{A}} u_\alpha = 0$, or, equivalently, if $\sum_{\alpha<d} u_\alpha = 1$, then $Q^{(d)}(v) = Q^{(d)}(u) = 1$. If d is odd, then $v \in \{\bar{e}_\alpha\}^{Q^{(d)}}_{\alpha\in\mathcal{A}}$ by induction hypothesis. If d is even and $p_{d-1}(u) = \bar{e}_\sharp$, then $d = 2 \bmod 4$ and we can use the previous lemma to obtain that $v \in \{\bar{e}_\alpha\}^{Q^{(d)}}_{\alpha\in\mathcal{A}}$. Otherwise, we can use the induction hypothesis to obtain the same result. In any case, $u = v + \bar{e}_d \in \{\bar{e}_\alpha\}^{Q^{(d)}}_{\alpha\in\mathcal{A}}$ by $Q^{(d)}$-closedness. Finally, assume that $\sum_{\alpha<d} u_\alpha = 0$, so $Q^{(d)}(v) = 0$.

Thus,

$$Q^{(d)}(v) = Q^{(d)}(u+\bar{e}_d) = Q^{(d)}(u) + Q^{(d)}(\bar{e}_d) + \langle u, \bar{e}_d\rangle = 0.$$

Since $u \notin \ker \overline{\Omega}$, there exists $\beta \in \mathcal{A}$ with $\langle u, \bar{e}_\beta\rangle = 1$ so, $Q^{(d)}(u+\bar{e}_\beta) = \langle u, \bar{e}_\beta\rangle = 1$. We know that $\beta < d$, since $Q^{(d)}(v) = 0$. Moreover, from $\overline{\Omega}_{\beta d} = 1$, we can see that:

$$Q^{(d)}(u+\bar{e}_\beta+\bar{e}_d) = Q^{(d)}(u+\bar{e}_\beta) + Q^{(d)}(\bar{e}_d) + \langle u+\bar{e}_\beta, \bar{e}_d\rangle = 1.$$

If d is odd, then $\overline{\Omega}^{(d-1)}$ is invertible, so $p_{d-1}(u+\bar{e}_\beta+\bar{e}_d) \notin \ker \overline{\Omega}^{(d-1)}$ and we conclude by induction hypothesis and $Q^{(d)}$-closedness since $(u+\bar{e}_\beta+\bar{e}_d)+(\bar{e}_\beta+\bar{e}_d) = u$. If, on the contrary, d is even, we conclude in the same way if $u+\bar{e}_\beta+\bar{e}_d \neq (\bar{e}_\sharp\ \ 0)$. Otherwise, $d = 2 \bmod 4$ and $u = \sum_{\alpha\neq\beta} \bar{e}_\alpha$, so we conclude by the previous lemma. □

We obtain the following proposition which completes the proof of Theorem 2.1.4:

Proposition 2.4.8. *For even d, the orthogonal transvections $\{\overline{T}_\alpha\}_{\alpha\in\mathcal{A}}$ generate the orthogonal group $O(Q^{(d)})$. In particular, it is contained in $\overline{\mathrm{RV}}(\pi^{(d)})$.*

Proof. By the previous lemma, we have that $\{e_\alpha\}_{\alpha\in\mathcal{A}}^{Q^{(d)}} = \mathrm{NS}(Q^{(d)})$. Since the dimension of $H_1(M^{(d)}\setminus\Sigma^{(d)}; \mathbb{Z}/2\mathbb{Z})$ is at least 6, the orthogonal transvections generate $O(Q^{(d)})$. We conclude by Lemma 2.4.4. □

2.5 Non-hyperelliptic connected components of $\mathcal{H}(g-1, g-1)$

As was done by Avila, Matheus and Yoccoz for the hyperelliptic components [AMY18], we can give an explicit description of the Rauzy–Veech groups for the remaining connected components of $\mathcal{H}(g-1, g-1)$. We will continue using the representatives found in the previous section. We will prove the following theorem:

Theorem 2.5.1. *For any $g \geq 3$, the Rauzy–Veech group of a nonhyperelliptic connected component of $\mathcal{H}(g-1, g-1)$ is equal to the preimage of the orthogonal group $\mathrm{O}(Q^{(d)})$ by the modulo-two reduction $\mathrm{Sp}(\Omega^{(d)}, \mathbb{Z}) \cap \mathrm{SL}(H_1(M^{(d)}\setminus\Sigma^{(d)})) \to \mathrm{Sp}(\overline{\Omega}^{(d)}, \mathbb{Z}/2\mathbb{Z})$, where $d = 2g+1$. If g is odd, it is isomorphic to $\mathrm{RV}(\pi^{(d-1)}) \ltimes \mathbb{Z}^{d-1}$. If g is even, it is isomorphic to a finite-index subgroup of $\mathrm{Sp}(\Omega^{(d-1)}, \mathbb{Z}) \ltimes \mathbb{Z}^{d-1}$. Moreover, such groups are generated by $\{T_\alpha\}_{\alpha\in\mathcal{A}}$.*

Assume for the rest of this section that $d \geq 7$ is odd. Let G_d be the preimage in the statement of the theorem. We have that $\mathrm{RV}(\pi^{(d)}) \subseteq G_d$ by Lemma 2.2.12. We will prove that every element of G_d belongs to $\mathrm{RV}(\pi^{(d)})$.

Recall that $e_\sharp = (1, -1, 1, \ldots, -1, 1)$ and that $\ker \Omega^{(d)}$ is generated by $e_\sharp$. Consider the $\mathbb{Z}$-submodule V of $H_1(M^{(d)}\setminus\Sigma^{(d)})$ spanned by $\{e_\alpha\}_{\alpha<d}$. Observe that $\Omega^{(d)}|_V$ is nondegenerate and that $V \oplus \ker \Omega^{(d)} = H_1(M^{(d)}\setminus\Sigma^{(d)})$. We have the following decomposition:

Lemma 2.5.2. *For any $S \in G_d$, we have that S acts as the identity on $\ker \Omega^{(d)}$. Moreover, if the maps $S^0 : V \to \ker \Omega^{(d)}$ and $S^1 : V \to V$ are the unique linear maps satisfying $S|_V = S^0 + S^1$, then $S^1 \in \mathrm{Sp}(\Omega^{(d)}|_V, \mathbb{Z})$.*

Proof. The fact that $S^1 \in \mathrm{Sp}(\Omega^{(d)}|_V, \mathbb{Z})$ follows from a straightforward computation:

$$\langle S^1(u), S^1(v)\rangle = \langle S^1(u) + S^0(u), S^1(v) + S^0(v)\rangle = \langle S(u), S(v)\rangle = \langle u, v\rangle$$

for any $u, v \in V$. Since $\Omega^{(d)}|_V$ is nondegenerate we obtain that $\det S^1 = 1$. We have that S preserves $\ker \Omega^{(d)}$. In other words, $e_\sharp$ is an eigenvector of S. In this way, we obtain that $1 = \det S = \det S^1 \det S|_{\ker \Omega^{(d)}}$. Thus, $\det S|_{\ker \Omega^{(d)}} = 1$, so $S|_{\ker \Omega^{(d)}} = \mathrm{Id}_{\ker \Omega^{(d)}}$. □

For any $S, T \in G_d$, we have that $(TS)^0 = T^0 S^1 + S^0$ and that $(TS)^1 = T^1 S^1$. In particular, $\{S^1 \mid S \in G_d\}$ is a subgroup of $\mathrm{Sp}(\Omega^{(d)}|_V, \mathbb{Z})$.

Lemma 2.5.3. *The group $\mathrm{O}(Q^{(d)})$ is isomorphic to $\mathrm{Sp}(\overline{\Omega}^{(d-1)}, \mathbb{Z}/2\mathbb{Z})$ if $d = 1 \bmod 4$ and to $\mathrm{O}(Q^{(d-1)}) \ltimes (\mathbb{Z}/2\mathbb{Z})^{d-1}$ if $d = 3 \bmod 4$.*

Proof. If $d = 1 \bmod 4$, then $Q^{(d)}(\bar{e}_\sharp) = 1$. Therefore, $Q^{(d)}$ is *regular*, that is, the only element of $\ker \Omega^{(d)} \cap \mathrm{S}(Q^{(d)})$ is 0. In this case, the map $S \mapsto S^1$ from $\mathrm{O}(Q^{(d)})$ to $\mathrm{Sp}(\overline{\Omega}^{(d)}|_{\overline{V}}, \mathbb{Z}/2\mathbb{Z})$ is an isomorphism [Gro02, Theorem 14.1, Theorem 14.2].

If $d = 3 \bmod 4$, then $Q^{(d)}(\bar{e}_\sharp) = 0$, so $Q^{(d)}$ is not regular. The restriction of $Q^{(d)}$ to $\overline{V}$ is a quadratic form for the nondegenerate symplectic form $\overline{\Omega}^{(d)}|_{\overline{V}}$. The natural projection $p_{d-1} : \overline{V} \to H_1(M^{(d-1)} \setminus \Sigma^{(d-1)}; \mathbb{Z}/2\mathbb{Z})$ to the first $d-1$ coordinates is an isomorphism between $Q^{(d)}|_{\overline{V}}$ and $Q^{(d-1)}$. In particular, the Arf invariants of $Q^{(d)}|_{\overline{V}}$ and $Q^{(d-1)}$ coincide. Observe that $S^1 \in \mathrm{O}(Q^{(d)}|_{\overline{V}})$ for any $S \in \mathrm{O}(Q^{(d)})$, since

$$\begin{aligned} Q^{(d)}(S^1(u)) &= Q^{(d)}(S^1(u) + S^0(u) + S^0(u)) = Q^{(d)}(S(u) + S^0(u)) \\ &= Q^{(d)}(S(u)) + Q^{(d)}(S^0(u)) + \langle S(u), S^0(u)\rangle = Q^{(d)}(u) \end{aligned}$$

for any $u \in \overline{V}$, so we obtain that $\{S^1 \mid S \in \mathrm{O}(Q^{(d)})\} \subseteq \mathrm{O}(Q^{(d)}|_{\overline{V}})$. By Lemma 2.4.7, we have that $\{p_{d-1}(\bar{e}_\alpha)\}_{\alpha<d}^{Q^{(d-1)}} = \mathrm{NS}(Q^{(d-1)})$ and therefore that $\{\bar{e}_\alpha\}_{\alpha<d}^{Q^{(d)}} = \overline{V} \cap \mathrm{NS}(Q^{(d)})$, showing that $\{S^1 \mid S \in \mathrm{O}(Q^{(d)})\} = \mathrm{O}(Q^{(d)}|_{\overline{V}})$, which is isomorphic to $\mathrm{O}(Q^{(d-1)})$.

Furthermore, for any $S^1 \in \mathrm{O}(Q|_{\overline{V}})$ we can choose any linear map $S^0 : \overline{V} \to \ker \overline{\Omega}^{(d)}$ and define S as the identity on $\ker \overline{\Omega}^{(d)}$ and as $S^0 + S^1$ on $\overline{V}$. We have that $S \in \mathrm{O}(Q^{(d)})$ since:

$$Q(S(u)) = Q(S^0(u) + S^1(u)) = Q(S^0(u)) + Q(S^1(u)) + \langle S^0(u), S^1(u)\rangle = Q(S^1(u)) = Q(u)$$

for every $u \in V$.

Finally, for every $S^0, T^0 : \overline{V} \to \ker \overline{\Omega}^{(d)}$ there exist unique elements v, w of $\overline{V}$ such that $S^0(u) = \langle u, v\rangle \bar{e}_\sharp$ and $T^0(u) = \langle u, w\rangle \bar{e}_\sharp$ for every $u \in \overline{V}$. We identify S^0 with v and T^0 with w.

From the equality $(TS)^0 = T^0S^1 + S^0$, we obtain that

$$(TS)^0(u) = \langle S^1(u), w\rangle\bar{e}_\sharp + \langle u, v\rangle\bar{e}_\sharp = \langle u, (S^1)^\intercal(w) + v\rangle\bar{e}_\sharp,$$

where $(S^1)^\intercal$ is the transpose of S^1 for the nondegenerate symplectic form $\overline{\Omega}^{(d)}|_{\overline{V}}$. This shows that $\mathrm{O}(Q^{(d)})$ is isomorphic to $\mathrm{O}(Q^{(d)}|_{\overline{V}}) \ltimes \overline{V}$ and to $\mathrm{O}(Q^{(d-1)}) \ltimes (\mathbb{Z}/2\mathbb{Z})^{d-1}$. □

The following two propositions complete the proof of Theorem 2.5.1.

Proposition 2.5.4. *For any even $g \geq 4$, the Rauzy–Veech group of $\mathcal{H}(g-1, g-1)$ is equal to G_d, where $d = 2g+1$. It is isomorphic to a subgroup of* $\mathrm{Sp}(\Omega^{(d-1)}, \mathbb{Z}) \ltimes \mathbb{Z}^{d-1}$ *of index 2^{2g} and it is generated by $\{T_\alpha\}_{\alpha\in\mathcal{A}}$.*

Proof. Let $S \in G_d$. We will prove that $S \in \mathrm{RV}(\pi^{(d)})$. First, we can find $T \in \mathrm{RV}(\pi^{(d)})$ such that $T^1 = S^1$. This can be done since $p_{d-1}\colon V \to H_1(M^{(d-1)} \setminus \Sigma^{(d-1)})$ is an isomorphism conjugating the action of $\{T^1 \mid T \in \mathrm{RV}(\pi^{(d)})\}$ and $\mathrm{Sp}(\Omega^{(d-1)}, \mathbb{Z})$. Indeed, by Lemma 2.2.7 and the results of the previous two sections, the action of the subgroup of $\{T^1 \mid T \in \mathrm{RV}(\pi^{(d)})\}$ generated by the maps $\{T^1_\alpha \mid \alpha < d\}$ is conjugated by p_{d-1} to $\mathrm{RV}(\pi^{(d-1)}) \subseteq \mathrm{Sp}(\Omega^{(d-1)}, \mathbb{Z})$. The group $\mathrm{RV}(\pi^{(d-1)})$ is maximal in $\mathrm{Sp}(\Omega^{(d-1)}, \mathbb{Z})$ by Remark 2.1.2. Since $\overline{T}^1_d$ does not preserve $Q^{(d)}|_{\overline{V}}$, we have that $(p_{d-1})_* T^1_d \notin \mathrm{RV}(\pi^{(d-1)})$, showing that $\{T^1 \mid T \in \mathrm{RV}(\pi^{(d)})\} = \mathrm{Sp}(\Omega^{(d)}|_V, \mathbb{Z})$.

Now observe that, for any $u, v \in V$,

$$\begin{aligned} T^{-2}_{v+e_\sharp}T^2_v(u) &= T^{-2}_{v+e_\sharp}(u + 2\langle v, u\rangle v) = u + 2\langle v, u\rangle v - 2\langle v + e_\sharp, u + 2\langle v, u\rangle v\rangle(v + e_\sharp) \\ &= u + \langle 2u, v\rangle e_\sharp. \end{aligned}$$

We set $S_v = T^{-2}_{v+e_\sharp}T^2_v$. Clearly $S^1_v = \mathrm{Id}|_V$, so $(S_vS_w)^1 = \mathrm{Id}|_V$ and $(S_vS_w)^0 = S^0_{v+w}$ for any $v, w \in V$.

We will now show that $S_v \in \mathrm{RV}(\pi^{(d)})$ for every $v \in V$. Indeed, we start by showing that $T^2_{e_\alpha+e_\sharp} \in \mathrm{RV}(\pi^{(d)})$ for every $\alpha < d$. By Lemma 2.4.2, we have that $w = \sum_{\alpha=1}^{d-3}(-1)^\alpha e_\alpha$ belongs to $\{e_\alpha\}^{\Omega^{(d)}}_{\alpha\in\mathcal{A}}$. We will consider several cases.

We can assume that $\pi^{(d)} = \tau^{(d)}$, since $\tau^{(d)}$ and $\sigma^{(d)}$ represent the same connected component. If $\alpha \leq d-3$, then $1 = \langle e_\alpha, e_{d-2}\rangle = \langle e_\alpha + e_{d-2}, w\rangle$, so $e_\alpha - w + e_{d-2} \in \{e_\alpha\}^{\Omega^{(d)}}_{\alpha\in\mathcal{A}}$. We can use Corollary 2.2.10 with $v'_1 = -e_{d-1} + e_d$, $v'_2 = e_\alpha - w + e_{d-2}$, $v'_3 = w$ and $v'_4 = e_d - (-1)^\beta e_\beta$, where $\beta = 4$ if $\alpha = 1$ and $\beta = 1$ if $\alpha > 1$. We obtain that $T^2_{e_\alpha+e_\sharp} \in \mathrm{RV}(\pi^{(d)})$ if $\alpha \leq d-3$. Now assume that $\alpha = d-2$. We have that

$$1 = \langle e_{d-2}, e_{d-1}\rangle = \langle e_{d-2} - e_{d-1}, e_{d-2}\rangle = \langle 2e_{d-2} - e_{d-1}, e_d\rangle,$$

so $2e_{d-2} - e_{d-1} + e_d \in \{e_\alpha\}^{\Omega^{(d)}}_{\alpha\in\mathcal{A}}$. We can use Corollary 2.2.10 with the following choices: $v'_1 = -w$, $v'_2 = 2e_{d-2} - e_{d-1} + e_d$, $v'_3 = e_d$ and $v'_4 = e_d + e_1$. Thus, we obtain that $T^2_{e_{d-2}+e_\sharp} \in \mathrm{RV}(\pi^{(d)})$. Finally, assume that $\alpha = e_{d-1}$. We can use Corollary 2.2.10 with $v'_1 = -w$, $v'_2 = e_{d-2} + e_d$, $v'_3 = e_d$ and

$v_4' = e_d + e_1$. We obtain that $T^2_{e_\alpha+e_\sharp} \in \mathrm{RV}(\pi^{(d)})$ for every $\alpha < d$, so $S_{e_\alpha} \in \mathrm{RV}(\pi^{(d)})$ for every $\alpha < d$.

By writing $v = \sum_{\alpha<d} n_\alpha e_\alpha$, we get that $S_v = S_{e_1}^{n_1} \cdots S_{e_{d-1}}^{n_{d-1}} \in \mathrm{RV}(\pi^{(d)})$ for every $v \in V$.

Finally, let $v, w \in V$ such that $S^0 = \langle \cdot, v\rangle e_\sharp$ and $T^0 = \langle \cdot, w\rangle e_\sharp$. Since $S^1 = T^1$, we have that $\overline{S}^0 = \overline{T}^0$ since the map $\mathrm{O}(Q^{(d)}) \to \mathrm{Sp}(\overline{\Omega}^{(d)}, \mathbb{Z}/2\mathbb{Z})$ is injective. We obtain that $v = w \bmod 2$, so there exists $u \in V$ such that $2u = v - w$. Moreover, observe that $(TS_u)^1 = T^1 = S^1$ and that $(TS_u)^0 = T^0 + S_u^0 = S^0$, so $S = TS_u \in \mathrm{RV}(\pi^{(d)})$.

The group G_d is isomorphic to a subgroup of $\mathrm{Sp}(\Omega^{(d)}|_V, \mathbb{Z}) \ltimes V$ that we will soon describe. Let $S^1 \in \mathrm{Sp}(\overline{\Omega}^{(d)}|_{\overline{V}}, \mathbb{Z}/2\mathbb{Z})$. There exists a unique linear map $S^0 \colon \overline{V} \to \ker \overline{\Omega}^{(d)}$ such that the map S defined as the identity on $\ker \overline{\Omega}^{(d)}$ and as $S^0 + S^1$ on $\overline{V}$ belongs to $\mathrm{O}(Q^{(d)})$. We can characterise S^0 as follows: $S^0(u)$ is the unique element of $\ker \Omega^{(d)}$ which satisfies the relation $Q^{(d)}(S^0(u)) = Q^{(d)}(S^1(u)) + Q^{(d)}(u)$ [Gro02, Theorem 14.1]. We obtain that:

$$S^0(u) = \begin{cases} 0 & Q^{(d)}(S^1(u)) = Q^{(d)}(u) \\ \bar{e}_\sharp & Q^{(d)}(S^1(u)) \neq Q^{(d)}(u). \end{cases}$$

If $S^1, T^1 \in \mathrm{Sp}(\overline{\Omega}^{(d)}|_{\overline{V}}, \mathbb{Z}/2\mathbb{Z})$, observe that $Q^{(d)}(S^1(u)) = Q^{(d)}(T^1(u))$ for every $u \in \overline{V}$ if and only if they belong to the same coset of $\mathrm{Sp}(\overline{\Omega}^{(d)}|_{\overline{V}}, \mathbb{Z}/2\mathbb{Z})/\mathrm{O}(Q^{(d)}|_{\overline{V}})$. Let $v_{S^1} \in \overline{V}$ be the unique element of $\overline{V}$ such that $S^0 = \langle \cdot, v_{S^1}\rangle \bar{e}_\sharp$. The image of the map $S^1 \mapsto v_{S^1}$ consists of $2^{g-1}(2^g \pm 1)$ elements, depending on the Arf invariant of $\mathrm{O}(Q^{(d)}|_{\overline{V}})$.

We can now describe G_d up to isomorphism as follows: for each $S^1 \in \mathrm{Sp}(\Omega^{(d)}|_V, \mathbb{Z})$, we define $V_{S^1} = \{w \in V \mid \overline{w} = v_{\bar{S}^1}\}$. Then, G_d is isomorphic to $\bigcup_{S^1 \in \mathrm{Sp}(\Omega^{(d)}|_V, \mathbb{Z})} \{S^1\} \times V_{S^1}$, regarded as subgroup of $\mathrm{Sp}(\Omega^{(d)}|_V, \mathbb{Z}) \ltimes V$. It has index 2^{2g} in $\mathrm{Sp}(\Omega^{(d)}|_V, \mathbb{Z}) \ltimes V$ since a coset C is determined by the unique vector $v \in \overline{V}$ such that $(\mathrm{Id}|_V, w) \in C$ for every $w \in V$ with $\overline{w} = v$. □

Proposition 2.5.5. *For any odd $g \geq 3$, the Rauzy–Veech group of $\mathcal{H}(g-1, g-1)$ is equal to G_d, where $d = 2g+1$. It is isomorphic to $\mathrm{RV}(\pi^{(d-1)}) \ltimes \mathbb{Z}^{d-1}$ and it is generated by $\{T_\alpha\}_{\alpha \in \mathcal{A}}$.*

Proof. The proof is very similar to that of Proposition 2.5.4. Let $S \in G_d$. We will prove that $S \in \mathrm{RV}(\pi^{(d)})$. First, we can find $T \in \mathrm{RV}(\pi^{(d)})$ such that $T^1 = S^1$. This can be done since $p_{d-1} \colon V \to H_1(M^{(d-1)} \setminus \Sigma^{(d-1)})$ is an isomorphism conjugating the action of $\{T^1 \mid T \in \mathrm{RV}(\pi^{(d)})\}$ and $\mathrm{RV}(\pi^{(d-1)})$ by Lemma 2.2.7 and our previous results.

Now observe that, for any $u, v \in V$,

$$T_{v+e_\sharp}^{-1} T_v(u) = T_{v+e_\sharp}^{-1}(u + \langle v, u\rangle v) = u + \langle v, u\rangle v - \langle v + e_\sharp, u + \langle v, u\rangle v\rangle (v + e_\sharp) = u + \langle u, v\rangle e_\sharp.$$

We set $S_v = T_{v+e_\sharp}^{-1} T_v$. Clearly $S_v^1 = \mathrm{Id}|_V$, so $(S_v S_w)^1 = \mathrm{Id}|_V$ and $(S_v S_w)^0 = S_{v+w}^0$ for any $v, w \in V$.

We will now show that $S_v \in \mathrm{RV}(\pi^{(d)})$ for every $v \in V$. Indeed, we start by showing that $e_\alpha + e_\sharp \in \{e_\alpha\}_{\alpha \in \mathcal{A}}^{\Omega^{(d)}}$ for every $\alpha < d$. Indeed, by Lemma 2.4.2, $w = \sum_{\alpha=1}^{d-1} (-1)^\alpha e_\alpha$ belongs to $\{e_\alpha\}_{\alpha \in \mathcal{A}}^{\Omega^{(d)}}$. Since $1 = \langle e_\alpha, e_d\rangle = \langle e_\alpha + e_d, w\rangle$, we obtain that $e_\alpha + e_d - w = e_\alpha + e_\sharp \in \{e_\alpha\}_{\alpha \in \mathcal{A}}^{\Omega^{(d)}}$.

We conclude that $S_{e_\alpha} \in \mathrm{RV}(\pi^{(d)})$ for every $\alpha < d$. By writing $v = \sum_{\alpha<d} n_\alpha e_\alpha$, we get that $S_v = S_{e_1}^{n_1} \cdots S_{e_{d-1}}^{n_{d-1}} \in \mathrm{RV}(\pi^{(d)})$ for every $v \in V$.

Finally, let $v, w \in V$ such that $S^0 = S_v^0$ and $T^0 = S_w^0$. Observe that $(TS_{v-w})^1 = T^1 = S^1$ and that $(TS_{v-w})^0 = T^0 + S_{v-w}^0 = S^0$, so $S = TS_{v-w} \in \mathrm{RV}(\pi^{(d)})$. □

2.6 Rauzy–Veech groups of general strata

We will define the Rauzy–Veech group in "absolute homology" by its action on the space $H_1(M_\pi \setminus \Sigma_\pi)/\ker \Omega_\pi$ and denote it by $\mathrm{RV}(\pi)|_H \subseteq \mathrm{Sp}(\Omega_\pi, \mathbb{Z})|_H$, where the latter group is the group of symplectic automorphisms on $H_1(M_\pi \setminus \Sigma_\pi)/\ker \Omega_\pi$. We have seen in the last chapter that the more classical definition by its action on $H(\pi)$ produces an isomorphic group.

Using the adjacency of strata, we will prove that $\mathrm{RV}(\pi)|_H$ contains the Rauzy–Veech group of a connected component of a minimal stratum. We will then analyse which strata are adjacent to each other, which will conclude the proof of Theorem 2.1.1 since the Rauzy–Veech group of any connected component of a minimal stratum is Zariski-dense by the classification of connected components [KZ03], the work on the hyperelliptic case by Avila, Matheus and Yoccoz [AMY18] and Theorem 2.1.4.

The following concepts were introduced by Avila and Viana [AV07b, Section 5]:

Definition 2.6.1. Let π be an irreducible permutation on an alphabet $\mathcal{A}$. Let $\alpha \in \mathcal{A}$ and let π' be the permutation on $\mathcal{A} \setminus \{\alpha\}$ obtained by erasing the letter α from the top and bottom rows of π. If π' is irreducible, we say that it is a *simple reduction* of π.

We also need a slightly stronger definition:

Definition 2.6.2. Let π' be an irreducible permutation on an alphabet $\mathcal{A}'$ not containing α. Let $\beta, \beta' \in \mathcal{A}'$ such that $(\alpha_{\mathrm{t},1}, \alpha_{\mathrm{b},1}) \neq (\beta, \beta')$. We define the permutation π on the alphabet $\mathcal{A}' \cup \{\alpha\}$ by inserting α just before β in the top row and just before β' in the bottom row of π'. We say that π is a *simple extension* of π'. Moreover, we say that such simple extension is *genus-preserving* if the genera of the surfaces $M_{\pi'}$ and M_π coincide.

This definition can be extended, by the same rule, to the Rauzy diagram of π'. We denote by $\mathscr{E}$ the *extension map* defined in this way.

If π is a simple extension of π', then π is also irreducible [AV07b, Lemma 5.4] and π' is a simple reduction of π. Conversely, if π' is a simple reduction of π whose omitted letter is not the last on the top or bottom row of π, then π is a simple extension of π'.

Simple extensions allow us to find copies of simpler Rauzy–Veech groups inside more complex ones. Indeed, we start by recalling the definition of the extension map. Let γ' be an arrow in the Rauzy diagram of π' starting at π'. We define a path $\mathscr{E}_*(\gamma')$ in the Rauzy diagram of π starting at π as follows: (1) if γ' is a top arrow and the letter α is added before the last letter on the bottom row of π', then $\mathscr{E}_*(\gamma')$ is constructed by applying two top operations to π; (2) if γ' is a bottom arrow and the letter α is added before the last letter on the top row of π', then $\mathcal{E}_*(\gamma')$ is constructed by applying two bottom operations to π; (3) otherwise, $\mathcal{E}_*(\gamma')$ is constructed by applying one operation of the same type as γ' to π. In every case, $\mathcal{E}_*(\gamma')$ starts at the image by $\mathcal{E}$ of the start of γ' and ends at the image by $\mathcal{E}$ of the end of γ'. This definition can be extended to any walk on the Rauzy diagram of π' by concatenation. We refer the reader to the work of Avila and Viana for more details [AV07b, Section 5.2].

We restate a lemma [AV07b, Lemma 5.6] from their work to match our context.

Lemma 2.6.3. *Let π be a genus-preserving simple extension of π'. Then, the canonical injection $\iota\colon H_1(M_{\pi'} \setminus \Sigma_{\pi'}) \to H_1(M_\pi \setminus \Sigma_\pi)$ descends to the quotients by $\ker \Omega_{\pi'}$ and $\ker \Omega_\pi$, respectively, into a symplectic isomorphism. Moreover, it conjugates the actions of $\mathrm{Sp}(\Omega_{\pi'}, \mathbb{Z})|_H$ and $\mathrm{Sp}(\Omega_\pi, \mathbb{Z})|_H$, and also the actions of $\mathrm{RV}(\pi')|_H$ and a subgroup of $\mathrm{RV}(\pi)|_H$.*

Proof. First observe that $H_1(M_{\pi'} \setminus \Sigma_{\pi'})/\ker \Omega_{\pi'}$ and $H_1(M_\pi \setminus \Sigma_\pi)/\ker \Omega_\pi$ have the same rank as the genus is preserved. Therefore, ι maps $\ker \Omega_{\pi'}$ into $\ker \Omega_\pi$ and descends to a well-defined symplectic isomorphism which we will also denote by ι.

Now, let γ' be an arrow in the Rauzy diagram of π' starting at π' and let $\gamma = \mathcal{E}_*(\gamma')$. Assume that γ is constructed as case (1) in the definition. Let β and β' be the last letters on the top and bottom rows of π', respectively. Then, by definition, $B_{\gamma'}^{-1} = \mathrm{Id} - E_{\beta'\beta}$ and $B_\gamma^{-1} = (\mathrm{Id} - E_{\beta'\beta})(\mathrm{Id} - E_{\alpha\beta}) = \mathrm{Id} - E_{\beta'\beta} - E_{\alpha\beta}$. From these relations, it is easy to see that $\iota(u)B_\gamma^{-1} = \iota(u)(\mathrm{Id} - E_{\beta'\beta})$ for every $u \in H_1(M_{\pi'} \setminus \Sigma_{\pi'})$, so $\iota(uB_{\gamma'}^{-1}) = \iota(u)B_\gamma^{-1}$ for every vector $u \in H_1(M_{\pi'} \setminus \Sigma_{\pi'})$. Similar computations for cases (2) and (3) show that ι_* is a monomorphism mapping $\mathrm{RV}(\pi')|_H$ to a subgroup of $\mathrm{RV}(\pi)|_H$. □

In particular, we obtain that $[\mathrm{Sp}(\Omega_\pi)|_H : \mathrm{RV}(\pi)|_H] \leq [\mathrm{Sp}(\Omega_{\pi'})|_H : \mathrm{RV}(\pi')|_H]$.

We also have that genus-preserving simple extensions preserve the spin parity:

Lemma 2.6.4. *Let π be a genus-preserving simple extension of π'. Then, the Arf invariants of Q_π and $Q_{\pi'}$ coincide.*

Proof. Let $\iota\colon H_1(M_{\pi'} \setminus \Sigma_{\pi'}) \to H_1(M_\pi \setminus \Sigma_\pi)$ be the canonical injection. Then, one has that $Q_{\pi'}(v) = Q_\pi(\iota(v))$ and $\langle v, w\rangle_{\pi'} = \langle \iota(v), \iota(w)\rangle_\pi$ for every $v, w \in H_1(M_{\pi'} \setminus \Sigma_{\pi'})$. Since the genus is preserved, we obtain that if $(v_\alpha, w_\alpha)_{\alpha\in\mathcal{B}}$ is a maximal symplectic subset of $H_1(M_{\pi'} \setminus \Sigma_{\pi'})$, then $(\iota(v_\alpha), \iota(w_\alpha))_{\alpha\in\mathcal{B}}$ is a maximal symplectic subset of $H_1(M_\pi \setminus \Sigma_\pi)$ and we conclude that

$$\mathrm{Arf}(Q_{\pi'}) = \sum_{\alpha\in\mathcal{B}} Q_\pi(v_\alpha)Q_\pi(w_\alpha) = \sum_{\alpha\in\mathcal{B}} Q_\pi(\iota(v_\alpha))Q_\pi(\iota(w_\alpha)) = \mathrm{Arf}(Q_\pi).$$

□

We say that a permutation π is *standard* if $\pi_{\mathrm{b}}(\alpha_{\mathrm{t},d}) = 1$ and $\pi_{\mathrm{b}}(\alpha_{\mathrm{t},1}) = d$. Observe that such a permutation is always irreducible. It was proven by Rauzy that standard permutations exist in every Rauzy class [Rau79]. The following lemma asserts the existence of some genus-preserving simple extensions for such permutations:

Lemma 2.6.5. *Let $\pi = (\pi_{\mathrm{t}}, \pi_{\mathrm{b}})$ be a standard permutation from $\mathcal{A}$ to $\{1, \ldots, d\}$. Assume that $M_\pi \in \mathcal{H}(m_1, \ldots, m_n)$ where $m_1 \geq 2$ and $m_i \geq 1$ for every $2 \leq i \leq n$. Then, there exists an irreducible permutation π' such that $M_{\pi'} \in \mathcal{H}(m_{1,1}, m_{1,2}, m_2, \ldots, m_n)$ and such that π' is a simple extension of π, where $m_{1,1}, m_{1,2} \geq 1$ are any integers satisfying $m_{1,1} + m_{1,2} = m_1$.*

Proof. Let $p\colon P_\pi \to M_\pi$ be the projection map obtained by identifying the sides of the polygon P_π by translation.

Consider the bijection $s\colon \mathcal{A} \times \{\mathrm{t}, \mathrm{b}\} \to \mathcal{A} \times \{\mathrm{t}, \mathrm{b}\}$ defined by:

- $s(\alpha_{\mathrm{t},j}, \mathrm{t}) = (\alpha_{\mathrm{t},j-1}, \mathrm{b})$ if $j > 1$;
- $s(\alpha_{\mathrm{t},1}, \mathrm{t}) = (\alpha_{\mathrm{t},d}, \mathrm{t})$;
- $s(\alpha_{\mathrm{b},j}, \mathrm{b}) = (\alpha_{\mathrm{b},j+1}, \mathrm{t})$ if $j < d$; and
- $s(\alpha_{\mathrm{b},d}, \mathrm{b}) = (\alpha_{\mathrm{b},1}, \mathrm{b})$.

The bijection s encodes the process of turning around a marked point in a clockwise manner. Indeed, the set $\mathcal{A} \times \{\mathrm{t}\}$ corresponds to the top sides of P_π, while the set $\mathcal{A} \times \{\mathrm{b}\}$ corresponds to the bottom sides. For $\alpha \in \mathcal{A}$, let $z \in P_\pi$ be its left endpoint. The orbit of (α, t) by s is equal to the set of top sides of P_π whose left endpoints z' satisfy $p(z') = p(z)$ and the bottom sides of P_π whose right endpoints z' satisfy $p(z') = p(z)$.

We can use the orbit by s to compute the conical angle of $p(z)$. Indeed, it is easy to see that such angle is equal to $\pi|\mathrm{Orb}_s(\alpha, \mathrm{t}) \setminus \{(\alpha_{\mathrm{t},1}, \mathrm{t}), (\alpha_{\mathrm{b},d}, \mathrm{b})\}|$.

Now, let $z \in \mathbb{C}$ be a top vertex of P_π such that $p(z)$ is the conical singularity of order m_1. By using that π is standard, we can assume that $z \neq 0, d$. Indeed, if $p(0) = p(d)$, then the top $\alpha_{\mathrm{t},1}$-side of P_π joins 0 and a vertex $z \in P_\pi \setminus \{0, d\}$ such that $p(z) = p(0) = p(d)$. Let $\alpha \in \mathcal{A}$ such that z is the left endpoint of the top α-side of P_π, which exists since $z \neq d$. We have that $\alpha \neq \alpha_{1,\mathrm{t}}$ since $z \neq 0$.

Consider the set $\mathrm{Orb}_s(\alpha, \mathrm{t}) \setminus \{(\alpha_{\mathrm{t},1}, \mathrm{t}), (\alpha_{\mathrm{b},d}, \mathrm{b})\}$, ordered by the order its elements occur when applying s to (α, t) iteratively. Observe that an element is at an odd position if and only if it corresponds to a top side. Moreover, observe that its cardinality is $2+2m_1 = 2+2m_{1,1}+2m_{1,2}$. Let (β, t) be the $(3 + 2m_{1,1})$-th element. We define the simple extension π' of π by inserting a letter $\alpha' \notin \mathcal{A}$ before α in the top row and before β in the bottom row. Let $\mathcal{A}' = \mathcal{A} \cup \{\alpha'\}$.

We will now prove that $M_{\pi'} \in \mathcal{H}(m_{1,1}, m_{1,2}, m_2, \ldots, m_n)$. Indeed, consider the bijection $s'\colon \mathcal{A}' \times \{\mathrm{t}, \mathrm{b}\} \to \mathcal{A}' \times \{\mathrm{t}, \mathrm{b}\}$ defined for π' in an analogous way as s for π. It is easy to see that:

- $s'(\alpha', \mathrm{t}) = s(\alpha, \mathrm{t})$;
- $s'(\alpha, \mathrm{t}) = (\alpha', \mathrm{b})$;
- $s'(\alpha', \mathrm{b}) = (\beta, \mathrm{t})$;
- if $\beta = \alpha_{\mathrm{b},1}$, then $s'(\alpha_{\mathrm{t},1}, \mathrm{t}) = (\alpha', \mathrm{t})$. Otherwise, let $\beta' \in \mathcal{A}$ be the letter before β in π_{b} and then $s'(\beta', \mathrm{b}) = (\alpha', \mathrm{t})$;

and that s' coincides with s elsewhere.

Let k be the smallest natural number satisfying $s^{k+1}(\alpha, \mathrm{t}) = (\beta, \mathrm{t})$. If $\beta = \alpha_{\mathrm{b},1}$, then we have that $s^k(\alpha, \mathrm{t}) = (\alpha_{\mathrm{t},1}, \mathrm{t})$ and, otherwise, $s^k(\alpha, \mathrm{t}) = (\beta', \mathrm{b})$. In both cases, $s'(s^k(\alpha, \mathrm{t})) = (\alpha', \mathrm{t})$.

Furthermore, since we have that $s'(\alpha', \mathrm{t}) = s(\alpha, \mathrm{t})$ we obtain that the orbit of (α', t) by s' is:

$$(\alpha', \mathrm{t}), s(\alpha, \mathrm{t}), s^2(\alpha, \mathrm{t}), \ldots, s^k(\alpha, \mathrm{t}),$$

so, by the choice of β, $|\mathrm{Orb}_{s'}(\alpha', \mathrm{t}) \setminus \{(\alpha_{\mathrm{t},1}, \mathrm{t}), (\alpha_{\mathrm{b},d}, \mathrm{b})\}| = 2 + 2m_{1,1}$.

On the other hand, let ℓ be the smallest natural number satisfying $s^{\ell+1}(\beta, \mathrm{t}) = (\alpha, \mathrm{t})$. The orbit of (α, t) by s' is:

$$(\alpha, \mathrm{t}), (\alpha', \mathrm{b}), (\beta, \mathrm{t}), s(\beta, \mathrm{t}), s^2(\beta, \mathrm{t}), \ldots, s^\ell(\beta, \mathrm{t}),$$

so $|\mathrm{Orb}_{s'}(\alpha, \mathrm{t}) \setminus \{(\alpha_{\mathrm{t},1}, \mathrm{t}), (\alpha_{\mathrm{b},d}, \mathrm{b})\}| = 2 + 2m_1 - (2 + 2m_{1,1}) + 2 = 2 + 2m_{1,2}$.

These two orbits are disjoint, so the α'-side of $M_{\pi'}$ joins two distinct conical singularities of orders $m_{1,1}$ and $m_{1,2}$. Since s' coincides with s outside of these orbits, the orders of the rest of the conical singularities are preserved. □

We obtain the following corollary:

Corollary 2.6.6. *Let π be a standard permutation such that M_π has genus g. We have that:*

- *If $g \geq 3$ and M_π belongs to a connected stratum, then* $\mathrm{RV}(\pi)|_H = \mathrm{Sp}(\Omega_\pi, \mathbb{Z})|_H$.
- *If M_π belongs to $\mathcal{H}(2m_1, \ldots, 2m_n)^{\mathrm{spin}}$, where* spin $\in$ {even, odd}, *$n \geq 2$ and $m_i \geq 1$ for every $1 \leq i \leq n$, then* $\mathrm{RV}(\pi)|_H$ *contains an isomorphic copy of the Rauzy–Veech group of $\mathcal{H}(2g-2)^{\mathrm{spin}}$ and has index at most $2^{g-1}(2^g \pm 1)$ inside its ambient symplectic group.*

Proof. Assume first that M_π belongs to a connected stratum. Let π' be a standard permutation representing some connected component of $\mathcal{H}(2g-2)$. By iterating the previous lemma and the fact that every Rauzy class contains standard permutations, there exists a sequence of simple extensions and Rauzy inductions taking π' to π. Therefore, we obtain that $\mathrm{RV}(\pi)|_H$ contains an isomorphic copy of $\mathrm{RV}(\pi')$. Recall that the Rauzy–Veech groups of nonhyperelliptic components are maximal subgroups of $\mathrm{Sp}(2g, \mathbb{Z})$ by Remark 2.1.2. If $g = 3$, the index of the Rauzy–Veech group of $\mathcal{H}(4)^{\mathrm{odd}}$ is 28, while the index of the Rauzy–Veech group of $\mathcal{H}(4)^{\mathrm{hyp}}$ is 288, which is not divisible by 28. Therefore, $\mathrm{RV}(\pi)|_H = \mathrm{Sp}(\Omega_\pi, \mathbb{Z})|_H$ in this case by maximality. If $g \geq 4$, $\mathrm{RV}(\pi)|_H$ contains isomorphic copies of the Rauzy–Veech groups of both nonhyperelliptic components of $\mathcal{H}(2g-2)$, so we also conclude by maximality as in Remark 2.1.2.

Now, if M_π belongs to $\mathcal{H}(2m_1, \ldots, 2m_n)^{\mathrm{spin}}$ let π' be a standard permutation representing $\mathcal{H}(2g-2)^{\mathrm{spin}}$. By iterating the previous lemma and using the fact that the Arf invariants of the quadratic forms are preserved by simple extensions, there exists a sequence of simple extensions and Rauzy inductions taking π' to π, which completes the proof. □

The following lemma completes our classification of the Rauzy–Veech groups:

Lemma 2.6.7. *The Rauzy–Veech group of $\mathcal{H}(2m_1, \ldots, 2m_n)^{\mathrm{spin}}$ in "absolute homology", where* spin $\in$ {even, odd}*, $n \geq 2$ and $m_i \geq 1$ for every $1 \leq i \leq n$, is isomorphic to the Rauzy–Veech group of $\mathcal{H}(2g-2)^{\mathrm{spin}}$ and has index $2^{g-1}(2^g \pm 1)$ inside its ambient symplectic group.*

Proof. We will use the explicit permutation representatives of such strata computed by Zorich [Zor08, Proposition 3, Proposition 4]. We start by proving the result for $\mathcal{H}(2, 2, \ldots, 2)^{\mathrm{odd}}$. Let $g \geq 3$ and consider the representative:

$$\tau = \begin{pmatrix} 0 & 1 & 2 & 3 & 4 & 5 & 6 & 7 & \cdots & 3g-7 & 3g-6 & 3g-5 & 3g-4 & 3g-3 \\ 3 & 2 & 4 & 6 & 5 & 7 & 9 & 8 & \cdots & 3g-5 & 3g-3 & 3g-4 & 1 & 0 \end{pmatrix}.$$

It is easy to see that $\{e_0 - e_1 + e_{3k+1}\}_{k=1}^{g-2}$ is a basis of $\ker \Omega_\tau$ consisting of singular vectors for Q_τ. Therefore, $\ker \Omega_\tau \subseteq \mathrm{S}(Q_\tau)$. Thus, Q_τ descends to $H_1(M_\tau \setminus \Sigma_\tau; \mathbb{Z}/2\mathbb{Z})/\ker \overline{\Omega}_\tau$ into a well-defined quadratic form which we will also denote by Q_τ. Now, observe that, for any $S \in \overline{\mathrm{RV}}(\tau)$ and any $u \in H_1(M_\tau \setminus \Sigma_\tau; \mathbb{Z}/2\mathbb{Z})$,

$$Q_\tau(S([u]) = Q_\tau([S(u)]) = Q_\tau(S(u)) = Q_\tau(u) = Q_\tau([u]).$$

We conclude that $\overline{\mathrm{RV}}(\tau)|_H$ preserves Q_τ and therefore that $\mathrm{RV}(\tau)|_H$ is isomorphic and has the same index as $\mathrm{RV}(\tau^{(2g)})$.

Now, a representative of any connected component of the form $\mathcal{H}(2m_1, \ldots, 2m_n)^{\mathrm{odd}}$ can be obtained by genus-preserving simple reductions of τ which consist of erasing some letters of the form $3k+1$ for $1 \leq k \leq g-2$. These simple reductions are also simple extensions when reversed, so the index of the Rauzy–Veech group of $\mathcal{H}(2, 2, \ldots, 2)^{\mathrm{odd}}$ cannot be larger than the index of the Rauzy–Veech group of $\mathcal{H}(2m_1, \ldots, 2m_n)^{\mathrm{odd}}$. We conclude by the previous corollary.

The proof for the even components is completely analogous by using the representative:

$$\sigma = \begin{pmatrix} 0 & 1 & 2 & 3 & 4 & 5 & 6 & 7 & \cdots & 3g-7 & 3g-6 & 3g-5 & 3g-4 & 3g-3 \\ 6 & 5 & 4 & 3 & 2 & 7 & 9 & 8 & \cdots & 3g-5 & 3g-3 & 3g-4 & 1 & 0 \end{pmatrix},$$

the basis $\{\sum_{\alpha=0}^{6}(-1)^\alpha e_\alpha\} \cup \{e_1 - e_2 + e_{3k+1}\}_{k=2}^{g-2}$ of $\ker \Omega_\sigma$ consisting of singular vectors for Q_σ. □

Recalling Theorem 2.1.4, Theorem 2.5.1 and Lemma 2.2.7 and combining our results with those of Avila, Matheus and Yoccoz [AMY18] we can summarise the full classification of the Rauzy–Veech groups:

Theorem 2.6.8. *For a connected component $\mathcal{C}$ of a stratum of genus-g translation surfaces, we denote by $\mathrm{RV}(\mathcal{C})$ its associated Rauzy–Veech group and by $\mathrm{RV}(\mathcal{C})|_H$ the restriction of $\mathrm{RV}(\mathcal{C})$ to "absolute homology". We have the following:*

- *If $\mathcal{C}$ is hyperelliptic, then the group $\mathrm{RV}(\mathcal{C})$ is characterised by preserving a specific finite set modulo two [AMY18, Theorem 2.9].*

- *The group* $\mathrm{RV}(\mathcal{H}(2m_1, \ldots, 2m_n)^{\mathrm{spin}})|_H$*, where* spin $\in$ {even, odd}*,* $n \geq 2$ *and* $m_i \geq 1$ *for every* $1 \leq i \leq n$*, is the subgroup of* $\mathrm{Sp}(2g, \mathbb{Z})$ *whose modulo-two reduction preserves a specific quadratic form whose Arf invariant is* spin*. Its index inside* $\mathrm{Sp}(2g, \mathbb{Z})$ *is* $2^{g-1}(2^g + 1)$ *for even spin and* $2^{g-1}(2^g - 1)$ *for odd spin.*
- *For any other connected component* $\mathcal{C}$*, including every connected stratum for* $g \geq 3$*, one has that* $\mathrm{RV}(\mathcal{C})|_H = \mathrm{Sp}(2g, \mathbb{Z})$.

Moreover, $\mathrm{RV}(\mathcal{C})|_H$ *is generated by the symplectic transvections along the canonical vectors.*

Chapter 3

Simplicity of the Lyapunov spectra of certain quadratic differentials

The purpose of this chapter is to prove the simplicity of the Lyapunov spectra of certain strata of quadratic differentials. It is an adapted version of the article "Simplicity of the Lyapunov spectra of certain quadratic differentials" [Gut17].

3.1 Introduction

The analogue of the Kontsevich–Zorich conjecture for half-translation surfaces is not known. Recall from Chapter 1 that there are two families of exponents, usually called "plus" and "minus" [EKZ14]. Some dynamical properties, such as the deviation of ergodic averages [For02; EKZ14] are described by the "minus" Lyapunov exponents, which are defined in terms of the canonical orientable double cover of the quadratic differential. Other dynamical properties, such as the diffusion rate of windtree models [DHL14; DZ15], are controlled by the "plus" Lyapunov exponents. It is known that the smallest nonnegative "plus" and "minus" exponents are positive and that the two largest "minus" exponents are distinct [Tre13]. However, the simplicity is not known for neither family in higher genus.

In this chapter we study the Rauzy–Veech groups of some strata of the moduli space of quadratic differentials. We restrict to the case of connected components of strata having at least three singularities and at least one of odd order. Observe that all such strata are connected except for some strata of the form $\mathcal{Q}(4j+2, 2k-1, 2k-1)$ or $\mathcal{Q}(2j-1, 2j-1, 2k-1, 2k-1)$ for integers $j, k \geq 0$ and the exceptional strata $\mathcal{Q}(6, 3, -1)$ and $\mathcal{Q}(3, 3, 3, -1)$ in genus 3, and $\mathcal{Q}(3, 3, 3, 3)$ and $\mathcal{Q}(6, 3, 3)$ in genus 4.

Our main result is the following:

Theorem 3.1.1. *The "plus" Rauzy–Veech group of any connected component of a stratum of meromorphic quadratic differentials defined on genus-g Riemann surfaces with $g \geq 1$ having at most simple poles and at least three singularities (zeros or poles), not all of even order, and the "minus" Rauzy–Veech group* *of connected components of strata satisfying the same conditions and having exactly two singularities of odd order are finite-index subgroups of their ambient symplectic groups. More precisely*:

- *The "plus" and "minus" Rauzy–Veech groups of* $\mathbb{Q}(4j+2, 2k-1, 2k-1)^{\mathrm{hyp}}$ *contain the Rauzy–Veech group of* $\mathcal{H}(2g-2)^{\mathrm{hyp}}$ *for every* $j, k \geq 0$.
- *The "plus" Rauzy–Veech group of* $\mathbb{Q}(2j-1, 2j-1, 2k-1, 2k-1)^{\mathrm{hyp}}$ *contains the Rauzy–Veech group of* $\mathcal{H}(g-1, g-1)^{\mathrm{hyp}}$ *for every* $j \geq 1$ *and* $k \geq 0$.
- *The "plus" and "minus" Rauzy–Veech groups of* $\mathbb{Q}(2, 3, 3)^{\mathrm{nonhyp}}$ *contain the Rauzy–Veech group of* $\mathcal{H}(4)^{\mathrm{odd}}$*, so their indices are at most* 28.
- *The "plus" Rauzy–Veech group of* $\mathbb{Q}(3, 3, -1, -1)^{\mathrm{nonhyp}}$ *is equal to its entire ambient symplectic group.*
- *In any other case, provided the conditions on the singularities are satisfied, the Rauzy–Veech groups are equal to their entire ambient symplectic groups, except if* $g = 2$ *where they contain the Rauzy–Veech group of* $\mathcal{H}(2)$ *and their indices are thus at most* 6.

Observe that most Rauzy–Veech groups of connected components of strata satisfying our hypotheses are equal to their entire ambient symplectic groups. The possible exceptions are hyperelliptic components and some other components in genus two and three.

To prove the Kontsevich–Zorich conjecture, Avila and Viana established a general criterion for the simplicity of the Lyapunov spectrum of symplectic cocycles [AV07a; AV07b]. In the case of the Teichmüller geodesic flow, this general criterion amounts to showing that the underlying monoid is *pinching* and *twisting*, which is almost automatic in the case that the group arising from the monoid is a finite-index subgroup of the symplectic group. Therefore, we obtain the following corollary:

Corollary 3.1.2. *The "plus" Lyapunov spectrum of any connected component of any stratum of meromorphic quadratic differentials defined on Riemann surfaces of genus at least one having at most simple poles and at least three singularities* (*zeros or poles*)*, not all of even order, is simple. Moreover, the "minus" Lyapunov spectrum of connected components of strata satisfying the same conditions and having exactly two singularities of odd order is also simple.*

In order to prove Theorem 3.1.1, we rely on the classification of Rauzy–Veech groups of Abelian differentials in the previous chapter. Indeed, we generalise the notion of simple extension defined by Avila and Viana [AV07b, Section 5.2] to find explicit combinatorial adjacencies between the strata mentioned in Theorem 3.1.1 and some strata of Abelian differentials. Then, we use the fact that the indices of the Rauzy–Veech groups of Abelian differentials are finite.

3.2 Permutations with involution

We will use some results of Avila and Resende's work [AR12], which uses a slightly different formalism for generalised permutations. Indeed, let $\mathcal{A}$ be an alphabet with $2d$ equipped with a fixed-point-free involution $\iota\colon \mathcal{A} \to \mathcal{A}$. Let $* \notin \mathcal{A}$ be a letter outside of the alphabet $\mathcal{A}$. We say that a bijection $\tau : \mathcal{A} \cup \{*\} \to \{1, \dots, 2d + 1\}$ is a *permutation with involution* if $\iota(\mathcal{A}_\mathrm{l}) \not\subseteq \mathcal{A}_\mathrm{r}$ and $\iota(\mathcal{A}_\mathrm{r}) \not\subseteq \mathcal{A}_\mathrm{l}$, where $\mathcal{A}_\mathrm{l} = \{\alpha \in \mathcal{A} \mid \tau(\alpha) < \tau(*)\}$ and $\mathcal{A}_\mathrm{r} = \{\alpha \in \mathcal{A} \mid \tau(\alpha) > \tau(*)\}$. We write such a bijection as a table

$$\tau = \begin{pmatrix} \tau^{-1}(1) & \tau^{-1}(2) & \dots & \tau^{-1}(2d+1) \end{pmatrix}.$$

For any generalised permutation π on an alphabet $\mathcal{A}$, we can define a permutation with involution τ on the alphabet $\mathcal{A} \times \{0, 1\}$ as follows: consider the involution $\iota\colon \mathcal{A} \times \{0, 1\} \to \mathcal{A} \times \{0, 1\}$ defined as $\iota(\alpha, \varepsilon) = (\alpha, 1 - \varepsilon)$ for each $\alpha \in \mathcal{A}$ and $\varepsilon \in \{0, 1\}$ and let

$$\tau = \begin{pmatrix} (\pi(\ell+m), \varepsilon_{\ell+m}) & \cdots & (\pi(\ell+1), \varepsilon_{\ell+1}) & * & (\pi(1), \varepsilon_1) & \dots & (\pi(\ell), \varepsilon_\ell) \end{pmatrix}.$$

where the $\varepsilon_j \in \{0, 1\}$ are chosen so $\varepsilon_j = 1 - \varepsilon_{\sigma(j)}$ for every $j \in \{1, \dots, 2d\}$. Then, Convention 1 is equivalent to $\iota(\mathcal{A}_\mathrm{l}) \not\subseteq \mathcal{A}_\mathrm{r}$ and $\iota(\mathcal{A}_\mathrm{r}) \not\subseteq \mathcal{A}_\mathrm{l}$, so we obtain a bijection between generalised permutations satisfying Convention 1 and permutations with involution (up to exchanging letters in the same orbit of ι). Moreover, a straightforward computation shows that the right and left operations on permutations with involution correspond, respectively, to top and bottom operations on generalised permutations.

3.3 Simple extensions

Definition 3.3.1. Let τ be an irreducible generalised permutation on an alphabet $\mathcal{B}$ not containing α. We say that a generalised permutation π on the alphabet $\mathcal{B} \cup \{\alpha\}$ is a *simple extension* of τ if τ is obtained from π by erasing the letter α and the following conditions hold:

- α is not at the end of any row of π;
- at least one occurrence of α is not at the beginning of a row of π.

Observe that any simple extension of a strict generalised permutation also satisfies Convention 1. On the other hand, a simple extension of a permutation satisfies this convention only when it is a permutation as well.

We have that irreducibility is preserved by simple extensions of strict generalised permutations:

Lemma 3.3.2. *Let π be a simple extension of a strict generalised permutation τ obtained by inserting a letter α. If τ is irreducible, then π is irreducible as well.*

Proof. We will prove that if π is reducible, then the letter α was inserted in some positions that are forbidden by definition of simple extension. Assume then that π is reducible, so let

$$\pi = \left(\begin{array}{c|c|c} A \cup B & *** & D \cup B \\ \hline A \cup C & *** & D \cup C \end{array}\right), \text{ with } A, B, C, D \text{ disjoint subsets of } \mathcal{A},$$

be a decomposition as in the definition of reducibility.

Let $A' = A \setminus \{\alpha\}$, $B' = B \setminus \{\alpha\}$, $C' = C \setminus \{\alpha\}$ and $D' = D \setminus \{\alpha\}$. We have that

$$\tau = \left(\begin{array}{c|c|c} A' \cup B' & **\,*' & D' \cup B' \\ \hline A' \cup C' & **\,*' & D' \cup C' \end{array}\right)$$

and that this decomposition does not satisfy the definition of reducibility. That is: at least one corner is empty; if there is exactly one empty corner, it is on the right; and, if there are exactly two empty corners, they are on different sides. Observe that, in particular, $\alpha \in A \cup B \cup C \cup D$ and that τ has more empty corners than π. Therefore, the set in $\{A, B, C, D\}$ containing α is actually equal to $\{\alpha\}$.

We consider two cases:

1. If $A = \{\alpha\}$, then the left corners of π are not empty. If both right corners of π were nonempty, then both right corners of τ would be nonempty as well, which is not possible. By definition of reducibility, it is not possible that exactly one right corner of π is empty. Therefore, both of its right corners are empty. We conclude that α was inserted at the beginning of both rows.
2. If $B = \{\alpha\}$, $C = \{\alpha\}$ or $D = \{\alpha\}$, then one of the right corners must be equal to $\{\alpha\}$ since, otherwise, τ would not have more empty corners than π. Therefore, α was inserted at the end of a row.

□

If π is a simple extension of τ and η is an arrow in the Rauzy class of τ, we define the path $\gamma = \mathscr{E}_*(\eta)$ in the Rauzy class of π starting at π as follows:

1. If η is of top type and α is the next-to-last letter in the bottom row of τ, then γ consists of two top arrows if the occurrences of α are not consecutive, and of three top arrows if they are.
2. If η is of bottom type and α is the next-to-last letter in the top row of τ, then γ consists of two bottom arrows if the occurrences of α are not consecutive, and of three bottom arrows if they are.
3. Otherwise, γ consists of a single arrow of the same type of η.

Lemma 3.3.3. *Using the previous notation, γ is well-defined and its end π' is a simple extension of the end τ' of η.*

Proof. We consider the three cases separately. In case (1), we may have that the winning letter $*$ of η occurs in both rows. If the occurrences of α are not consecutive, then

$$\eta = \begin{pmatrix} \cdot & \cdot & \cdot & \cdot & * \\ \cdot & * & \cdot & \cdot & \beta \end{pmatrix} \to \begin{pmatrix} \cdot & \cdot & \cdot & \cdot & * \\ \cdot & * & \beta & \cdot & \cdot \end{pmatrix}$$

$$\gamma = \begin{pmatrix} \cdot & \cdot & \cdot & \cdot & \cdot & * \\ \cdot & * & \cdot & \cdot & \alpha & \beta \end{pmatrix} \to \begin{pmatrix} \cdot & \cdot & \cdot & \cdot & \cdot & * \\ \cdot & * & \beta & \cdot & \cdot & \alpha \end{pmatrix} \to \begin{pmatrix} \cdot & \cdot & \cdot & \cdot & \cdot & * \\ \cdot & * & \alpha & \beta & \cdot & \cdot \end{pmatrix}$$

Contrarily, if they are consecutive, then

$$\eta = \begin{pmatrix} \cdot & \cdot & \cdot & \cdot & * \\ \cdot & * & \cdot & \cdot & \beta \end{pmatrix} \rightarrow \begin{pmatrix} \cdot & \cdot & \cdot & \cdot & * \\ \cdot & * & \beta & \cdot & \cdot \end{pmatrix}$$

$$\begin{aligned} \gamma = \begin{pmatrix} \cdot & \cdot & \cdot & \cdot & \cdot & * \\ \cdot & * & \cdot & \alpha & \alpha & \beta \end{pmatrix} &\rightarrow \begin{pmatrix} \cdot & \cdot & \cdot & \cdot & \cdot & * \\ \cdot & * & \beta & \cdot & \alpha & \alpha \end{pmatrix} \\ \rightarrow \begin{pmatrix} \cdot & \cdot & \cdot & \cdot & \cdot & * \\ \cdot & * & \alpha & \beta & \cdot & \alpha \end{pmatrix} &\rightarrow \begin{pmatrix} \cdot & \cdot & \cdot & \cdot & \cdot & * \\ \cdot & * & \alpha & \alpha & \beta & \cdot \end{pmatrix} \end{aligned}$$

In both of these cases, the arrows exist since $\beta \neq *$ and $\alpha \neq *$.

Otherwise, both occurrences of $*$ are in the top row. If the occurrences of α are not consecutive, then

$$\eta = \begin{pmatrix} \cdot & \cdot & * & \cdot & * \\ \cdot & \cdot & \cdot & \cdot & \beta \end{pmatrix} \rightarrow \begin{pmatrix} \cdot & \cdot & \beta & * & \cdot & * \\ & \cdot & \cdot & \cdot & \cdot & \end{pmatrix}$$

$$\gamma = \begin{pmatrix} \cdot & \cdot & * & \cdot & \cdot & * \\ \cdot & \cdot & \cdot & \cdot & \alpha & \beta \end{pmatrix} \rightarrow \begin{pmatrix} \cdot & \cdot & \beta & * & \cdot & \cdot & * \\ & \cdot & \cdot & \cdot & \cdot & \alpha & \end{pmatrix} \rightarrow \begin{pmatrix} \cdot & \cdot & \beta & \alpha & * & \cdot & \cdot & * \\ & & \cdot & \cdot & \cdot & \cdot & & \end{pmatrix}$$

Contrarily, if they are consecutive, then

$$\eta = \begin{pmatrix} \cdot & \cdot & * & \cdot & * \\ \cdot & \cdot & \cdot & \cdot & \beta \end{pmatrix} \rightarrow \begin{pmatrix} \cdot & \cdot & \beta & * & \cdot & * \\ & \cdot & \cdot & \cdot & \cdot & \end{pmatrix}$$

$$\begin{aligned} \gamma = \begin{pmatrix} \cdot & \cdot & * & \cdot & \cdot & * \\ \cdot & \cdot & \cdot & \alpha & \alpha & \beta \end{pmatrix} &\rightarrow \begin{pmatrix} \cdot & \cdot & \beta & * & \cdot & \cdot & * \\ & \cdot & \cdot & \cdot & \alpha & \alpha & \end{pmatrix} \\ \rightarrow \begin{pmatrix} \cdot & \cdot & \beta & \alpha & * & \cdot & \cdot & * \\ & & \cdot & \cdot & \cdot & \alpha & & \end{pmatrix} &\rightarrow \begin{pmatrix} \cdot & \cdot & \beta & \alpha & \alpha & * & \cdot & \cdot & * \\ & & & \cdot & \cdot & \cdot & & & \end{pmatrix} \end{aligned}$$

and, in both cases, the arrows exist since, as a top arrow starts at τ, there exists a duplicate letter in the bottom row of τ which is different from β (and α).

Observe that, in all cases, α cannot be at the end of a row of π' and at least one of its occurrences cannot be at the beginning of a row. Thus, π' is a simple extension of τ'.

Case (2) is completely analogous to case (1), so we will now prove case (3). If η is of top type and the winning letter $*$ occurs on both rows,

$$\eta = \begin{pmatrix} \cdot & \cdot & \cdot & \cdot & * \\ \cdot & * & \cdot & \cdot & \beta \end{pmatrix} \rightarrow \begin{pmatrix} \cdot & \cdot & \cdot & \cdot & * \\ \cdot & * & \beta & \cdot & \cdot \end{pmatrix}$$

$$\gamma = \begin{pmatrix} \cdot & \cdot & \alpha & \cdot & \cdot & * \\ \cdot & * & \cdot & \alpha & \cdot & \beta \end{pmatrix} \rightarrow \begin{pmatrix} \cdot & \cdot & \alpha & \cdot & \cdot & * \\ \cdot & * & \beta & \cdot & \alpha & \cdot \end{pmatrix}$$

and the arrow exists because $\beta \neq *$.

Oh the other hand, if $*$ occurs twice in the top row

$$\eta = \begin{pmatrix} \cdot & \cdot & * & \cdot & * \\ \cdot & \cdot & \cdot & \cdot & \beta \end{pmatrix} \to \begin{pmatrix} \cdot & \cdot & \beta & * & \cdot & * \\ & \cdot & \cdot & \cdot & \cdot & \end{pmatrix}$$

$$\gamma = \begin{pmatrix} \cdot & \alpha & \cdot & * & \cdot & * \\ \cdot & \cdot & \alpha & \cdot & \cdot & \beta \end{pmatrix} \to \begin{pmatrix} \cdot & \alpha & \cdot & \beta & * & \cdot & * \\ & \cdot & \cdot & \alpha & \cdot & \cdot & \end{pmatrix}$$

and the arrow exists since, as a top arrow starts at τ, there exists a duplicate letter in the bottom row of τ which is different from β.

Observe that, in both cases, α cannot be at the end of a row of π', since α is not the next-to-last letter in the bottom row. Thus, π' is a simple extension of τ'.

If η is an arrow of bottom type, the computations are analogous. □

Remark 3.3.4. When a sequence of simple extensions is clear from context, we will also call $\mathscr{E}_*$ the *composition* of the extension maps of these simple extensions.

3.4 Rauzy–Veech groups

3.4.1 The "plus" Rauzy–Veech group

Let $\mathscr{R}$ be a Rauzy diagram of generalised permutations. We consider an undirected version $\widetilde{\mathscr{R}}$ of $\mathscr{R}$: for each arrow $\gamma = \pi \to \pi'$ we add a reversed arrow $\gamma^{-1} = \pi' \to \pi$ and we define $B_{\gamma^{-1}} = B_\gamma^{-1}$. Now consider a walk $\gamma = \gamma_1\gamma_2\cdots\gamma_n$ in $\widetilde{\mathscr{R}}$ starting at π and ending at π'. We define $B_\gamma = B_{\gamma_n}B_{\gamma_{n-1}}\cdots B_{\gamma_1}$, which satisfies $\Omega_{\pi'} = B_\gamma\Omega_\pi B_\gamma^{\intercal}$. In particular, if $\pi' = \pi$ (that is, if γ is a cycle), one has that B_γ (acting on *row* vectors) belongs to $\mathrm{Sp}(\Omega_\pi, \mathbb{Z})$. The "plus" Rauzy–Veech group of π is the group generated by matrices of this form:

Definition 3.4.1. Let $\mathscr{R}$ be a Rauzy class and $\pi \in \mathscr{R}$ be a fixed vertex. We define the *"plus" Rauzy–Veech group* $\mathrm{RV}^+(\pi)$ as the set of matrices of the form $B_\gamma \in \mathrm{Sp}(\Omega_\pi, \mathbb{Z})$ where γ is a cycle on $\widetilde{\mathscr{R}}$ with endpoints at π. We will always consider the action of $\mathrm{RV}^+(\pi)$ on row vectors.

Observe if π, π' are vertices of the same Rauzy class $\mathscr{R}$, then $\mathrm{RV}^+(\pi)$ and $\mathrm{RV}^+(\pi')$ are isomorphic, so we can define the "plus" Rauzy–Veech group of a Rauzy class. Indeed, if γ is any walk joining π and π', then the conjugation by B_γ is an isomorphism between $\mathrm{Sp}(\Omega_\pi, \mathbb{Z})$ and $\mathrm{Sp}(\Omega_{\pi'}, \mathbb{Z})$ and between $\mathrm{RV}^+(\pi)$ and $\mathrm{RV}^+(\pi')$. This shows, in particular, that the "plus" Rauzy–Veech group of a Rauzy class has a well-defined index inside its ambient symplectic group.

In general, "plus" Rauzy–Veech groups are symplectic with respect to a degenerate symplectic form Ω_π. Therefore, we define the "plus" Rauzy–Veech group in "absolute homology" by its action on $H_1(M_\pi \setminus \Sigma_\pi)/\ker\Omega_\pi$. We denote this group by $\mathrm{RV}^+(\pi)|_H \le \mathrm{Sp}(\Omega_\pi, \mathbb{Z})|_H$, where the latter is the group of symplectic automorphisms on $H_1(M_\pi \setminus \Sigma_\pi)/\ker\Omega_\pi$. The previous the action on $V^+(\pi)$ produces an isomorphic group, as discussed in Chapter 1.

As it is also the case for strata of Abelian differentials, simple extensions allow us to find copies of simpler "plus" Rauzy–Veech groups inside more complex ones:

Lemma 3.4.2. *Let π be a genus-preserving simple extension of τ obtained by inserting a letter α. Then, the canonical injection $\iota\colon H_1(M_\tau \setminus \Sigma_\tau) \to H_1(M_\pi \setminus \Sigma_\pi)$ descends to the quotients by $\ker \Omega_\tau$ and $\ker \Omega_\pi$, respectively, into a symplectic isomorphism. Moreover, it conjugates the actions of $\mathrm{Sp}(\Omega_\tau, \mathbb{Z})|_H$ and $\mathrm{Sp}(\Omega_\pi, \mathbb{Z})|_H$, and also the actions of $\mathrm{RV}^+(\tau)|_H$ and a subgroup of $\mathrm{RV}^+(\pi)|_H$.*

Proof. First observe that $H_1(M_\tau \setminus \Sigma_\tau)/\ker \Omega_\tau$ and $H_1(M_\pi \setminus \Sigma_\pi)/\ker \Omega_\pi$ have the same rank as the genus is preserved. Therefore, ι maps $\ker \Omega_\tau$ into $\ker \Omega_\pi$ and descends to a well-defined symplectic isomorphism which we will also denote by ι.

Now, let η be an arrow in the Rauzy class of τ starting at τ and let $\gamma = \mathscr{E}_*(\eta)$. Assume that γ is constructed as case (1) in the definition. Let β and β' be the last letters in the top and bottom rows of τ, respectively. We have several possible cases for the values of B_γ^{-1} depending on whether $(\Omega_\pi)_{\beta'\beta} \neq 0$, $(\Omega_\pi)_{\alpha\beta} \neq 0$ and whether the occurrences of α are consecutive in π, which are listed below:

$$\begin{aligned}
(\mathrm{Id} - E_{\beta'\beta})(\mathrm{Id} - E_{\alpha\beta}) &= (\mathrm{Id} - E_{\beta'\beta}) - E_{\alpha\beta}\\
(\mathrm{Id} - E_{\beta'\beta})(\mathrm{Id} - E_{\alpha\beta} - 2E_{\alpha\alpha}) &= (\mathrm{Id} - E_{\beta'\beta}) + (-E_{\alpha\beta} - 2E_{\alpha\alpha})\\
(\mathrm{Id} - E_{\beta'\beta} - 2E_{\beta'\beta'})(\mathrm{Id} - E_{\alpha\beta}) &= (\mathrm{Id} - E_{\beta'\beta} - 2E_{\beta'\beta'}) - E_{\alpha\beta}\\
(\mathrm{Id} - E_{\beta'\beta} - 2E_{\beta'\beta'})(\mathrm{Id} - E_{\alpha\beta} - 2E_{\alpha\alpha}) &= (\mathrm{Id} - E_{\beta'\beta} - 2E_{\beta'\beta'}) + (-E_{\alpha\beta} - 2E_{\alpha\alpha}),\\
(\mathrm{Id} - E_{\beta'\beta})(\mathrm{Id} - E_{\alpha\beta})^2 &= (\mathrm{Id} - E_{\beta'\beta}) - 2E_{\alpha\beta}\\
(\mathrm{Id} - E_{\beta'\beta})(\mathrm{Id} - E_{\alpha\beta} - 2E_{\alpha\alpha})^2 &= \mathrm{Id} - E_{\beta'\beta}\\
(\mathrm{Id} - E_{\beta'\beta} - 2E_{\beta'\beta'})(\mathrm{Id} - E_{\alpha\beta})^2 &= (\mathrm{Id} - E_{\beta'\beta} - 2E_{\beta'\beta'}) - 2E_{\alpha\beta}\\
(\mathrm{Id} - E_{\beta'\beta} - 2E_{\beta'\beta'})(\mathrm{Id} - E_{\alpha\beta} - 2E_{\alpha\alpha})^2 &= \mathrm{Id} - E_{\beta'\beta} - 2E_{\beta'\beta'},
\end{aligned}$$

where we used that the three letters α, β' and β are distinct. From these relations, it is easy to see that, in any case, $\iota(uB_\eta^{-1}) = \iota(u)B_\gamma^{-1}$ for every $u \in H_1(M_\tau \setminus \Sigma_\tau)$. Indeed, one has that $\iota(v)E_{\alpha\beta} = \iota(v)E_{\alpha\alpha} = 0$ for any $v \in H_1(M_\tau \setminus \Sigma_\tau)$ as its α-coordinate is 0. We obtain that:

$$\iota(u)B_\gamma^{-1} = \begin{cases} \iota(u)(\mathrm{Id} - E_{\beta'\beta}) & \langle \theta_\beta, \theta_{\beta'} \rangle \neq 0 \\ \iota(u)(\mathrm{Id} - E_{\beta'\beta} - 2E_{\beta'\beta'}) & \langle \theta_\beta, \theta_{\beta'} \rangle = 0. \end{cases}$$

On the other hand, by definition,

$$uB_\eta^{-1} = \begin{cases} u(\mathrm{Id} - E_{\beta'\beta}) & \langle \theta_\beta, \theta_{\beta'} \rangle \neq 0 \\ u(\mathrm{Id} - E_{\beta'\beta} - 2E_{\beta'\beta'}) & \langle \theta_\beta, \theta_{\beta'} \rangle = 0, \end{cases}$$

so $\iota(uB_\eta^{-1}) = \iota(u)B_\gamma^{-1}$ as the α-coordinate of $\iota(uB_\eta^{-1})$ is also 0.

Similar computations for the remaining cases (2) and (3) show that ι_* is a monomorphism mapping $\mathrm{RV}^+(\tau)|_H$ to a subgroup of $\mathrm{RV}^+(\pi)|_H$. □

3.4.2 The "minus" Rauzy–Veech group

A similar construction can be used to define the "minus" Rauzy–Veech group encoding the action of the Rauzy–Veech algorithm in the homology group of a double cover. We will give a *partial* explicit definition of such group which will be enough to prove our results.

Recall the double cover construction from Chapter 1: $\widetilde{M}_\pi$ is the translation surface with marked points $\widetilde{\Sigma}_\pi$ and a projection $p\colon \widetilde{M}_\pi \to M_\pi$ induced by the involution $\iota\colon \widetilde{M}_\pi \to \widetilde{M}_\pi$.

The involution ι induces a splitting $H_1(\tilde{M}_\pi \setminus \tilde{\Sigma}_\pi) = H^+(\pi) \oplus H^-(\pi)$. Let $\mathcal{A}_{\mathrm{tb}} \subseteq \mathcal{A}$ be the set of letters occurring in both rows of π. For $\alpha \in \mathcal{A}_{\mathrm{tb}}$, the cycle $\theta_\alpha \in H_1(M_\pi \setminus \Sigma_\pi)$ lifts to two possible cycles $\tilde{\theta}^0_\alpha, \tilde{\theta}^1_\alpha \in H_1(\tilde{M}_\pi \setminus \tilde{\Sigma}_\pi)$, which can be though of as belonging to P^0_π and P^1_π, respectively. We define an alternate form indexed by $\mathcal{A}_{\mathrm{tb}}$ as:

$$(\tilde{\Omega}_\pi)_{\alpha\beta} = \begin{cases} +2 & i_\alpha < i_\beta \text{ and } j_\alpha > j_\beta \\ -2 & i_\beta < i_\alpha \text{ and } j_\beta > j_\alpha \\ 0 & \text{otherwise,} \end{cases}$$

where $i_\alpha < j_\alpha$ and $i_\beta < j_\beta$ are the positions of α and β in π, as above. Clearly, we have that $(\tilde{\Omega}_\pi)_{\alpha\beta} = 2(\Omega_\pi)_{\alpha\beta}$. This matrix is the intersection form of the cycles $\{\tilde{\theta}_\alpha\}_{\alpha\in\mathcal{A}_{\mathrm{tb}}}$, defined as $\tilde{\theta}_\alpha = \tilde{\theta}^0_\alpha - \tilde{\theta}^1_\alpha$, which belong to $H^-(\pi)$. Let $S^-(\pi)$ be the subspace of $H^-(\pi)$ generated by the cycles $\{\tilde{\theta}_\alpha\}_{\alpha\in\mathcal{A}_{\mathrm{tb}}}$.

Assume that the generalised permutation π is obtained by a sequence of simple extensions starting at the permutation τ. By definition of the extension map $\mathscr{E}_*$, a duplicate letter is never the winner of arrows in $\mathscr{E}_*(\eta)$, where η is an arrow on the Rauzy class of τ. For this reason, the set of letters occurring in both rows remains the constant while traversing these arrows. We will give a partial definition of the "minus" Rauzy–Veech group, in the sense that it does not encode the entire action on $H^-(\pi)$, but rather its action on the subspace $S^-(\pi)$.

As for the "plus" case, let $\widetilde{\mathcal{R}}$ be the undirected Rauzy class of π. Let $\gamma = \pi \to \pi'$ be an arrow in $\mathcal{R}$ and let α_{w} and α_{l} be, respectively, the winner and loser of the operation sending π to π'. We assume that $\alpha_{\mathrm{w}} \notin \mathcal{A}_{\mathrm{tb}}$ as discussed above. We define the *"minus" Kontsevich–Zorich matrix* indexed by $\mathcal{A}_{\mathrm{tb}} \times \mathcal{A}_{\mathrm{tb}}$ as the change of basis matrix mapping $\{\tilde{\theta}'_\alpha\}_{\alpha\in\mathcal{A}}$ to $\{\tilde{\theta}_\alpha\}_{\alpha\in\mathcal{A}}$, where $\{\tilde{\theta}'_\alpha\}_{\alpha\in\mathcal{A}}$ for $M_{\pi'}$ is defined in an analogous way as $\{\theta_\alpha\}_{\alpha\in\mathcal{A}}$ for M_π. The resulting matrices are:

$$\tilde{B}_\gamma = \begin{cases} \mathrm{Id} + E_{\alpha_{\mathrm{l}}\alpha_{\mathrm{w}}} & \alpha_{\mathrm{l}} \in \mathcal{A}_{\mathrm{tb}} \\ \mathrm{Id} & \alpha_{\mathrm{l}} \notin \mathcal{A}_{\mathrm{tb}} \end{cases}$$

See Figure 3.1 for some of the computations. We extend this definition for reversed arrows and walks to define the "minus" Rauzy–Veech group:

Definition 3.4.3. Let $\mathcal{R}$ be a Rauzy class and $\pi \in \mathcal{R}$ be a fixed vertex. We define the *"minus" Rauzy–Veech group* $\mathrm{RV}^-(\pi)$ as the set of matrices of the form $\tilde{B}_\gamma \in \mathrm{Sp}(\tilde{\Omega}_\pi, \mathbb{Z})$ where γ is a cycle

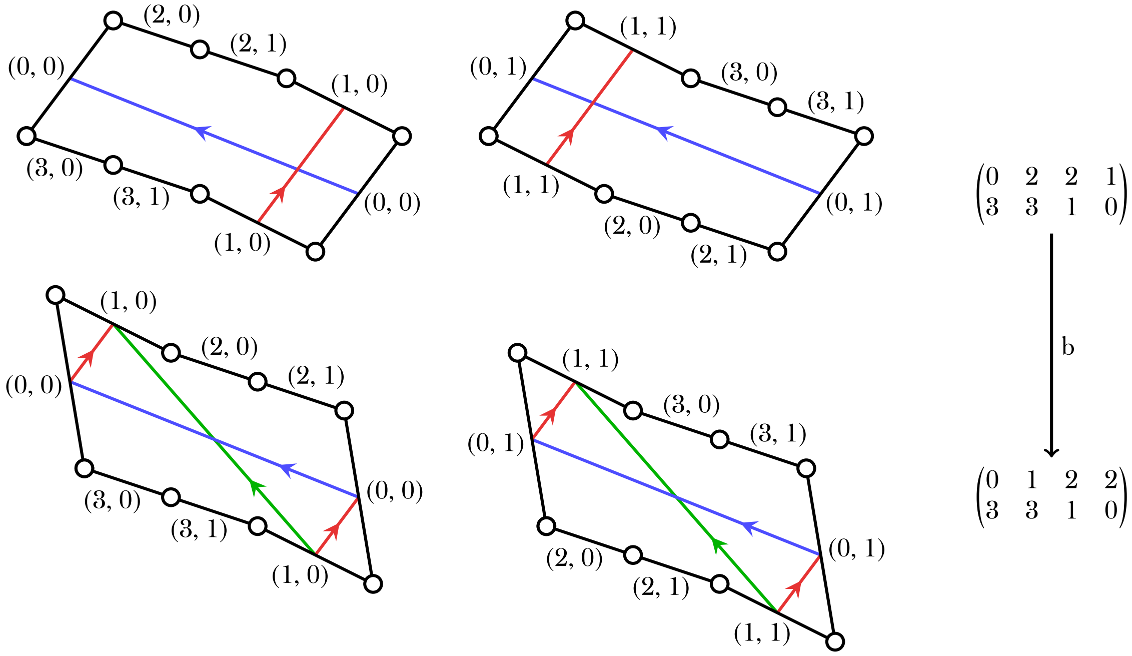


Figure 3.1: Some examples of the double cover construction and a Rauzy operation on it. The red lines represent $\tilde{\theta}_{\alpha_1}$, the blue lines $\tilde{\theta}_{\alpha_w} = \tilde{\theta}'_{\alpha_w}$ and the green lines $\tilde{\theta}'_{\alpha_1}$. When the winning letter occurs in both rows, the map $S^-(\pi') \to S^-(\pi)$ is represented in these bases as $\mathrm{Id} + E_{\alpha_1\alpha_w}$ if the losing letter belongs to $\mathcal{A}_{\mathrm{tb}}$.

on $\widetilde{\mathcal{R}}$ with endpoints at π such that no duplicate letter is the winner of an arrow of γ. We will always consider the action of $\mathrm{RV}^-(\pi)$ on row vectors.

Remark 3.4.4. We have defined the "minus" Rauzy–Veech group $\mathrm{RV}^-(\pi)$ restricted to the subspace $S^-(\pi)$ of $H_1(\tilde{M}_\pi \setminus \tilde{\Sigma}_\pi)$ explicitly. We can also define its action on $H_1(\tilde{M}_\pi \setminus \tilde{\Sigma}_\pi)$ in the same way, but we will not use explicit bases or matrices. Nevertheless, in analogy with the "plus" case, it is clear that this group preserves an intersection form and that it is well-defined for a Rauzy class. We will see below that, in some cases, our partial definition is sufficient to prove that the index of the "minus" Rauzy–Veech group is finite inside its ambient symplectic group.

Let $\mathcal{S}$ be a connected component of a stratum of quadratic differentials. Assume that there exists a generalised permutation π representing $\mathcal{S}$ obtained by genus-preserving simple extensions from a permutation τ.

If M_π belongs to the stratum $\mathcal{Q}(2m_1 - 1, \ldots, 2m_s - 1, 2m_{s+1}, \ldots, 2m_n)$, then it is well-known that $\tilde{M}_\pi$ belongs to the stratum $\mathcal{H}(2m_1, \ldots, 2m_s, m_{s+1}, m_{s+1}, \ldots, m_n, m_n)$. Thus, if g is the genus of M_π and $\tilde{g}$ is the genus of $\tilde{M}_\pi$, we have that

$$2\tilde{g} - 2 = 2\sum_{j=1}^{n} m_j = 2\sum_{j=1}^{n} m_j - s + s = 4g - 4 + s, \text{ so } \tilde{g} = 2g - 1 + \frac{s}{2}.$$

Let $q\colon H_1(\tilde{M}_\pi \setminus \tilde{\Sigma}_\pi) \to H_1(\tilde{M}_\pi)$ be the map obtained by "forgetting the punctures". If M_π has exactly two singularities of odd order, then $\tilde{g} = 2g$. Since $\dim H_1(\tilde{M}_\pi) = 2\tilde{g}$, we have that $\dim q(H_1^+(\pi)) = \dim q(H_1^-(\pi)) = 2g$. We conclude that the simple extensions taking τ to π are "doubly genus-preserving".

By hypothesis, we have that the rank of Ω_τ is $2g$. Moreover, since the alphabet $\mathcal{B}$ of τ is contained in $\mathcal{A}_{\mathrm{tb}}$ and $(\tilde{\Omega}_\pi)_{\alpha\beta} = 2(\Omega_\pi)_{\alpha\beta}$ for every $\alpha, \beta \in \mathcal{B}$, we have that the rank of $\tilde{\Omega}_\pi$ is also $2g$. Therefore, in analogy with the Abelian and the "plus" cases, $\tilde{\Omega}_\pi$ descends to the quotient $S^-(\pi)/\ker\tilde{\Omega}_\pi$ into a nondegenerate symplectic form. We denote the group of symplectic automorphisms of this space by $\mathrm{Sp}(\tilde{\Omega}_\pi, \mathbb{Z})|_H$. Let $\mathrm{RV}^-(\pi)|_H \leq \mathrm{Sp}(\tilde{\Omega}_\pi, \mathbb{Z})|_H$ be the group induced by the right action of $\mathrm{RV}^-(\pi)$ on this quotient. We can replicate the proof of Lemma 3.4.2 to conclude that it is enough to find simple extensions as above to obtain that the index of $\mathrm{RV}^-(\pi)|_H$ is finite in its ambient symplectic group $\mathrm{Sp}(\tilde{\Omega}_\pi, \mathbb{Z})|_H$.

3.5 Proof of the main theorem

By the discussion in the previous section, it is enough to find genus-preserving simple extensions starting at Abelian strata and ending at the desired connected components. The proof is divided in four cases: connected strata, hyperelliptic components, nonhyperelliptic components and exceptional strata. From now on, we will omit the words "plus" and "minus" when speaking of a Rauzy–Veech group, since the arguments work in both cases (provided that the relevant hypotheses are satisfied).

3.5.1 Connected strata

We will start by showing that simple extensions of generalised permutations that are not permutations allow us to "break up" the singularities of the meromorphic quadratic differential in any possible way. This lemma is analogous to the case of Abelian differentials [Gut19b, Lemma 6.5] and will allows us to obtain the proof of Theorem 3.1.1 for connected strata.

Lemma 3.5.1. *Let τ be a generalised permutation $\tau\colon \{1, \dots, 2d\} \to \mathcal{A}$. Assume that the surface $M_\tau \in \mathcal{Q}(m_1, \dots, m_n)$ where $m_1 \geq 1$ and $m_i \geq -1$ for every $2 \leq i \leq n$. Then, there exists an irreducible generalised permutation π such that $M_\pi \in \mathcal{Q}(m_{1,1}, m_{1,2}, m_2, \dots, m_n)$ and such that π is a simple extension of τ, where $m_{1,1}, m_{1,2} \geq -1$ are any integers satisfying $m_{1,1} + m_{1,2} = m_1$.*

Proof. Let $p\colon P_\tau \to M_\tau$ be the projection map obtained by identifying the sides of the polygon P_τ by translation and rotation.

Consider the bijection $s\colon \{1, \dots, 2d\} \to \{1, \dots, 2d\}$ defined by:

- $s(k) = \sigma(k-1)$ if $1 < k \leq \ell$;
- $s(1) = \sigma(\ell + 1)$;
- $s(k) = \sigma(k+1)$ if $\ell + 1 \leq k < \ell + m$; and
- $s(\ell + m) = \sigma(\ell)$.

The bijection s encodes the process of turning around a marked point in a clockwise manner. Indeed, the set $\{1,\dots,\ell\}$ corresponds to the top sides of P_τ, while the set $\{\ell+1,\dots,\ell+m\}$ corresponds to the bottom sides. For $1 \le k \le \ell$, let $z \in P_\tau$ be its left endpoint. The orbit of k by s is equal to the set of top sides of P_τ whose left endpoints z' satisfy $p(z') = p(z)$ and the bottom sides of P_τ whose right endpoints z' satisfy $p(z') = p(z)$.

We can use the orbit by s to compute the conical angle of $p(z)$. Indeed, it is easy to see that such angle is equal to $\pi|\mathrm{Orb}_s(k) \setminus \{1, \ell+m\}|$.

Now, let $z \in \mathbb{C}$ be a vertex of P_τ such that $p(z)$ is the conical singularity of order m_1. We can assume that $z \neq 0, e_\tau$. Indeed, let k be a top side whose left vertex is z or a bottom side whose right endpoint is z. Since $m_1 \ge 2$, one has that $|\mathrm{Orb}_s(k) \setminus \{1, \ell+m\}| \ge 3$, so there exists $1 < j \le \ell$ whose left vertex z' satisfies $p(z') = p(z)$ or $\ell+1 \le j < \ell+m$ whose right vertex z' satisfies $p(z') = p(z)$. We replace z by z' if necessary. For the rest of the proof, we will assume that $2 \le j \le \ell$, since the other case is analogous.

Consider the set $\mathrm{Orb}_s(j) \setminus \{1, \ell+m\}$, ordered by the order its elements occur when applying s to j iteratively. Its cardinality is equal to $2+m_1 = 2+m_{1,1}+m_{1,2}$. We will consider two cases:

If the $(2+m_{1,1})$-th element is in the bottom row, then the proof is very similar to the Abelian case. Nevertheless, we will present it in full detail for completeness. Let $\sigma(i) \neq j$ be the $(3+m_{1,1})$-th element. For a letter $\alpha' \notin \mathcal{A}$, we define the simple extension π of τ by inserting the letter α' just before the letters at positions i and j. That is, for every $k \in \{1,\dots,2(d+1)\}$,

$$\pi(k) = \begin{cases} \tau(k) & k < \min\{i,j\} \\ \alpha' & k = \min\{i,j\} \\ \tau(k-1) & \min\{i,j\} < k \le \max\{i,j\} \\ \alpha' & k = \max\{i,j\}+1 \\ \tau(k-2) & \max\{i,j\}+1 < k \le \ell+m+2 \end{cases}$$

and that the type of π is $(\ell', m') = (\ell+1, m+1)$.

Let $\iota\colon \{1,\dots,2d\} \to \{1,\dots,2(d+1)\}$ be the order-preserving injective map with $\pi \circ \iota = \tau$. Let i', j' be the positions of α' in π. We choose them so $i' < j'$ if and only if $i < j$.

We will now prove that $M_\pi \in \mathcal{Q}(m_{1,1}, m_{1,2}, m_2, \dots, m_n)$. Let σ' be the involution associated with π. Consider the bijection $s'\colon \{1,\dots,2(d+1)\} \to \{1,\dots,2(d+1)\}$ defined for π in an analogous way as s for τ. We have that:

- $s'(j') = \iota(s(j))$;
- $s'(j'+1) = i'$;
- $s'(i') = \iota(\sigma(i))$;
- $s'(1) = j'$ if $i' = \ell+1$ and $s'(i'-1) = j'$ otherwise;

and $s'(\iota(k)) = \iota(s(k))$ for any other $k \in \{1,\dots,2d\}$.

Let k be the smallest natural number satisfying $s^{k+1}(j) = \sigma(i)$. If $i = \ell+1$, then $s^k(j) = 1$ and, otherwise, $s^k(j) = i-1$. In any case, $s'(\iota(s^k(j))) = j'$. Moreover, since we have that $s'(j') = \iota(s(j))$

we obtain that the orbit of j' by s' is:

$$j', \iota(s(j)), \iota(s^2(j)), \dots, \iota(s^k(j)),$$

so, by the choice of i, $|\mathrm{Orb}_{s'}(j') \setminus \{1, \ell' + m'\}| = 2 + m_{1,1}$.

On the other hand, let k be the smallest natural number satisfying $s^{k+1}(\sigma(i)) = j$. The orbit of $j' + 1$ by s' is:

$$j' + 1, i', \iota(\sigma(i)), \iota(s(\sigma(i))), \iota(s^2(\sigma(i))), \dots, \iota(s^k(\sigma(i))),$$

so $|\mathrm{Orb}_{s'}(j' + 1) \setminus \{1, \ell' + m'\}| = 2 + m_1 - (2 + m_{1,1}) + 2 = 2 + m_{1,2}$.

These two orbits are disjoint, so the j'-side of M_π joins two distinct conical singularities of orders $m_{1,1}$ and $m_{1,2}$. Since s' coincides with s outside of these orbits, the orders of the rest of the conical singularities are preserved.

Otherwise, if the $(2 + m_{1,1})$-th element is in the top row, let i be such element (which may be equal to j). For a letter $\alpha' \notin \mathcal{A}$, we define the simple extension π of τ by inserting the letter α' just before the letters at positions i and j if $i \neq j$, and twice before the letter at position $i = j$ otherwise. The type of this generalised permutation is $(\ell', m') = (\ell + 2, m)$

Let $\iota\colon \{1, \dots, \ell + m\} \to \{1, \dots, \ell + m + 2\}$ be the order-preserving injective map such that $\pi \circ \iota = \tau$. Let $i' \neq j'$ be the positions of α' in π. We choose them so $i' < j'$ if and only if $i < j$ (in particular, if $i = j$ then $j' < i'$).

We will now prove that $M_\pi \in \mathcal{Q}(m_{1,1}, m_{1,2}, m_2, \dots, m_n)$. Let σ' be the involution associated with π. Consider the bijection $s'\colon \{1, \dots, 2(d + 1)\} \to \{1, \dots, 2(d + 1)\}$ defined for π in an analogous way as s for τ. We have that:

- $s'(j') = \iota(s(j))$;
- $s'(j' + 1) = i'$;
- $s'(i') = \iota(s(i))$ if $i \neq j$ and $s'(i') = i'$ otherwise;
- $s'(i' + 1) = j'$;

and $s'(\iota(k)) = \iota(s(k))$ for any other $k \in \{1, \dots, \ell + m\}$.

Let k be the smallest natural number satisfying $s^k(j) = i$. Observe that $\iota(i) = i' + 1$, so the orbit of j' by s' is:

$$j', \iota(s(j)), \iota(s^2(j)), \dots, \iota(s^k(j)) = i' + 1,$$

so, by the choice of β, $|\mathrm{Orb}_{s'}(j') \setminus \{1, \ell' + m'\}| = 2 + m_{1,1}$.

If $i \neq j$, let k be the smallest natural number satisfying $s^k(i) = j$. Since $\iota(j) = j' + 1$, the orbit of i' by s' is:

$$i', \iota(s(i)), \iota(s^2(i)), \dots, \iota(s^k(i)) = j' + 1,$$

so $|\mathrm{Orb}_{s'}(i') \setminus \{1, \ell' + m'\}| = 2 + m_1 - (2 + m_{1,1}) + 2 = 2 + m_{1,2}$.

Otherwise, we have that if $i = j$ and $m_{1,2} = -1$. In this case, the orbit of i' by s' consists only of i'. We obtain that $|\mathrm{Orb}_{s'}(i') \setminus \{1, \ell' + m'\}| = 1 = 2 + m_{1,2}$ as desired. □

Observe that the proof of the previous lemma can also be applied to a (genuine) permutation. Nevertheless, as explained after Definition 3.3.1, the resulting simple extension will not satisfy Convention 1 unless it is a permutation as well. This is related to the fact that there exist some global obstructions to “breaking up” some zeros of even order into two zeros of odd order using local operations [Bai+19]. This problem can be solved by using the previous lemma two times:

Corollary 3.5.2. *Let τ be a (genuine) permutation $\pi\colon \{1, \dots, 2d\} \to \mathcal{A}$. Assume that the surface $M_\tau \in \mathcal{H}(m_1, \dots, m_n)$ where $m_1 \geq 1$ and $m_i \geq 1$ for every $2 \leq i \leq n$. Then, there exists a generalised permutation π such that $M_\pi \in \mathcal{Q}(m_{1,1}, m_{1,2}, m_{1,3}, 2m_2, \dots, 2m_n)$ and such that π is a simple extension of τ, where $m_{1,1}, m_{1,2}, m_{1,3} \geq -1$ are any integers satisfying $m_{1,1}+m_{1,2}+m_{1,3} = 2m_1$ and $m_{1,1}, m_{1,2}$ are odd.*

Proof. We use the previous lemma two times, first adding a duplicate letter in the top row to split the conical singularity whose angle is $(2 + 2m_1)\pi$ into two conical singularities of angles $(2+m_{1,1})\pi$ and $(2+m_{1,2}+m_{1,3})\pi$. This is possible since, as $m_{1,1}$ is odd, the resulting generalised permutation cannot be a permutation. However, this generalised permutation does not satisfy Convention 1.

Now, we add a duplicate letter in the bottom row to split the conical singularity of angle $(2 + m_{1,2} + m_{1,3})\pi$ into two conical singularities of angle $(2 + m_{1,2})\pi$ and $(2 + m_{1,3})\pi$. Once again, this is possible because $m_{1,2}$ is odd. We obtain a permutation having a duplicate letter in each row, so it satisfies Convention 1. □

Remark 3.5.3. The strict generalised permutations produced by the previous corollary can be chosen to be irreducible. Indeed, because of the way we count the conical angle in the proof of Lemma 3.5.1, no letter is inserted at the beginning of a row. Therefore, we can construct a suspension datum $(\zeta'_\alpha)_{\alpha\in\mathcal{A}}$ for π starting from the “canonical” suspension datum $(\zeta_\alpha)_{\alpha\in\mathcal{B}}$ for τ by setting $\zeta'_\alpha = 1$ if $\alpha \in \mathcal{A} \setminus \mathcal{B}$.

Furthermore, even if the statements of the previous lemma and corollary forbid marked points which are not singularities, it is easy to see that the proof extends to the particular case of the permutation on a two-letter alphabet representing a torus with one marked point. This allows us to treat the genus-1 case below.

With these elements, the proof of Theorem 3.1.1 is a straightforward consequence of the classification of Rauzy–Veech groups for Abelian strata [AMY18; Gut19b]:

Proof of Theorem 3.1.1 for connected strata. Let $\mathcal{S}$ be a connected stratum of quadratic differentials of genus g with at least three singularities (zeros or poles), not all of even order. Clearly, there exist at least two singularities of odd order. Therefore, by the previous lemma and the previous corollary, there exists a sequence of simple extensions starting from any connected component of a minimal stratum of Abelian differentials and ending at $\mathcal{S}$. Indeed, we can first use the corollary once to create two singularities of odd order and then use the lemma iteratively to create the remaining singularities.

If $g \geq 4$, then the Rauzy–Veech group of $\mathcal{S}$ contains the Rauzy–Veech groups of both nonhyperelliptic minimal strata of genus-g Abelian differentials. These groups are maximal subgroups of $\mathrm{Sp}(2g, \mathbb{Z})$ [BGP14, Theorem 3] [1], so we obtain that the Rauzy–Veech group of $\mathcal{S}$ is $\mathrm{Sp}(2g, \mathbb{Z})$. If $g = 3$, then we can use the same argument as before, but using the connected components $\mathcal{H}(4)^{\mathrm{odd}}$ and $\mathcal{H}(4)^{\mathrm{hyp}}$. The Rauzy–Veech groups of such components have index 28 and 288, respectively. Since 28 does not divide 288, we conclude that the Rauzy–Veech group of $\mathcal{S}$ is $\mathrm{Sp}(6, \mathbb{Z})$ by maximality. If $g = 2$, we do not conclude that the Rauzy–Veech group is the entire ambient symplectic group, but we have that it has finite index because it contains a copy of the Rauzy–Veech group of $\mathcal{H}(2)$, which has index 6 inside $\mathrm{Sp}(4, \mathbb{Z})$. Finally, if $g = 1$, we can use the adjacency with the stratum $\mathcal{H}(0)$. Indeed, its Rauzy–Veech group is equal to its entire ambient symplectic group, so the same is true for the Rauzy–Veech group of $\mathcal{S}$. □

3.5.2 Hyperelliptic components

As was shown by Lanneau [Lan04], strata of the form $\mathcal{Q}(4j + 2, 2k - 1, 2k - 1)$ or of the form $\mathcal{Q}(2j - 1, 2j - 1, 2k - 1, 2k - 1)$ for any integers $j, k \geq 0$ have a hyperelliptic component. The genera of these strata are $j+k+1$ and $j+k$, respectively, and we will assume it to be at least one. We can find an explicit simple extension starting from a hyperelliptic connected component of Abelian strata, therefore showing that the desired Rauzy–Veech groups have finite index. Indeed, for an integer $d \geq 2$ consider the symmetric permutation on d letters:

$$\tau_d = \begin{pmatrix} 0 & 1 & 2 & \cdots & d-1 \\ d-1 & d-2 & d-3 & \cdots & 1 \end{pmatrix}.$$

Let $g_d = d/2$ if d is even and $(d - 1)/2$ if d is odd. It is well-known that this permutation represents the component $\mathcal{H}(2g_d - 2)^{\mathrm{hyp}}$ if d is even and $\mathcal{H}(g_d - 1, g_d - 1)^{\mathrm{hyp}}$ if d is odd.

Now, for $s, r \geq 1$ let

$$\sigma_{s,r} = \begin{pmatrix} 0 & A & 1 & 2 & \cdots & s & A & s+1 & s+2 & \cdots & s+r \\ s+r & \cdots & s+2 & s+1 & B & s & \cdots & 2 & 1 & B & 0 \end{pmatrix}$$

which can be obtained from τ_{s+r+1} by applying two simple extensions. By putting $s = 2k$ and $r = 2j + 1$, $\sigma_{s,r}$ represents the connected component $\mathcal{Q}(4j + 2, 2k - 1, 2k - 1)^{\mathrm{hyp}}$, and, by putting $s = 2k$ and $r = 2j$, it represents $\mathcal{Q}(2j-1, 2j-1, 2k-1, 2k-1)^{\mathrm{hyp}}$ [Zor08, Section 3.6]. In the former case, removing the letters A and B produces $\tau_{2j+2k+2}$, whose genus is $j + k + 1$. In the latter case, it produces $\tau_{2j+2k+1}$, whose genus is $j + k$. Therefore, we obtain genus-preserving simple extensions starting at hyperelliptic components of Abelian strata and ending at $\mathcal{Q}(4j + 2, 2k - 1, 2k - 1)^{\mathrm{hyp}}$ or $\mathcal{Q}(2j - 1, 2j - 1, 2k - 1, 2k - 1)^{\mathrm{hyp}}$. We conclude that the

[1] This result uses the classification of finite simple groups.

indices of their Rauzy–Veech groups are finite in their ambient symplectic groups.

3.5.3 Non-hyperelliptic components

All strata of the form $\mathbb{Q}(4j+2, 2k-1, 2k-1)$ and $\mathbb{Q}(2j-1, 2j-1, 2k-1, 2k-1)$, for integers $j, k \geq 0$ and genus at least two, have exactly two connected components, except for $\mathbb{Q}(6, 3, 3)$ and $\mathbb{Q}(3, 3, 3, 3)$ (which have three), and $\mathbb{Q}(2, 1, 1)$ and $\mathbb{Q}(1, 1, 1, 1)$ (which are connected). We will consider strata of this form with exactly two connected components and prove that the Rauzy–Veech group of the nonhyperelliptic one has finite index.

We will first prove it for strata of surfaces of genus at least three. Consider the following permutation representatives of minimal Abelian strata computed by Zorich:

$$\tau_g = \begin{pmatrix} 0 & 1 & 2 & 3 & 5 & 6 & \cdots & 3g-7 & 3g-6 & 3g-4 & 3g-3 \\ 3 & 2 & 6 & 5 & 9 & 8 & \cdots & 3g-3 & 3g-4 & 1 & 0 \end{pmatrix}$$

for $g \geq 3$ and

$$\sigma_g = \begin{pmatrix} 0 & 1 & 2 & 3 & 5 & 6 & \cdots & 3g-7 & 3g-6 & 3g-4 & 3g-3 \\ 6 & 5 & 3 & 2 & 9 & 8 & \cdots & 3g-3 & 3g-4 & 1 & 0 \end{pmatrix}$$

One has that τ_g represents the family of strata $\mathcal{H}(2g-2)^{\mathrm{odd}}$ for every $g \geq 3$ and that σ_g represents the family $\mathcal{H}(2g-2)^{\mathrm{even}}$ for every $g \geq 4$ [Zor08, Proposition 3, Proposition 4]. Moreover, these representatives have a single cylinder. One can use Lemma 3.5.1 and Corollary 3.5.2 to find representatives of $\mathbb{Q}(4j+2, 2k-1, 2k-1)$ and $\mathbb{Q}(2j-1, 2j-1, 2k-1, 2k-1)$, which will also have a single cylinder since, as is made explicit in the proof of Lemma 3.5.1, we can assume that the letters were not inserted at the beginning of the top row. We will show that such extensions are not hyperelliptic by using the following fact: a generalised permutation such that the first letter in its top row coincides with the last letter in its bottom row represents a hyperelliptic connected component if and only if the generalised permutation obtained by removing said letter has, up to cyclic permutations on both rows, one of the following two forms:

$$\begin{pmatrix} A & 1 & 2 & \cdots & s & A & s+1 & s+2 & \cdots & s+r \\ s+r & \cdots & s+2 & s+1 & B & s & \cdots & 2 & 1 & B \end{pmatrix}$$

or

$$\begin{pmatrix} 1 & 2 & \cdots & r+1 & 1 & 2 & \cdots & r+1 \\ r+2 & r+3 & \cdots & r+s+2 & r+2 & r+3 & \cdots & r+s+2 \end{pmatrix}.$$

This fact was proven by Lanneau [Zor08, Proposition 11]. It is clear that by inserting two or three letters to τ_g or σ_g it will not have one of those forms, so the extensions lie in the nonhyperelliptic components.

If the genus is three, we get sequences of simple extensions starting from $\mathcal{H}(4)^{\mathrm{odd}}$, whose Rauzy–Veech group has index 28 inside its ambient symplectic group, and ending at any nonexceptional and nonhyperelliptic component. Moreover, we can also find simple extensions starting from $\mathcal{H}(4)^{\mathrm{hyp}}$ and ending at the four components $\mathbb{Q}(10, -1, -1)^{\mathrm{nonhyp}}$, $\mathbb{Q}(6, 1, 1)^{\mathrm{nonhyp}}$,

$\mathbb{Q}(5,5,-1,-1)^{\text{nonhyp}}$, $\mathbb{Q}(3,3,1,1)^{\text{nonhyp}}$. Indeed, the generalised permutations

$$\begin{pmatrix} 1 & A & A & 2 & 3 & 4 & 5 & 6 \\ 6 & B & B & 5 & 4 & 3 & 2 & 1 \end{pmatrix} \quad \text{and} \quad \begin{pmatrix} 1 & A & 2 & 3 & A & 4 & 5 & 6 \\ 6 & B & 5 & 4 & B & 3 & 2 & 1 \end{pmatrix}$$

represent the components $\mathbb{Q}(10,-1,-1)^{\text{nonhyp}}$, $\mathbb{Q}(6,1,1)^{\text{nonhyp}}$, respectively, and erasing the letters A and B produces permutations representing $\mathcal{H}(4)^{\text{hyp}}$. The other two components can be obtained by using these two generalised permutations together with Lemma 3.5.1. Since the index of $\mathcal{H}(4)^{\text{hyp}}$ is 288, we obtain that the Rauzy–Veech groups of these four components are equal to their entire ambient symplectic groups by maximality.

There is only one remaining nonexceptional and nonhyperelliptic component in genus three: $\mathbb{Q}(2,3,3)^{\text{nonhyp}}$. In this case, we were not able to find a simple extension starting at $\mathcal{H}(4)^{\text{hyp}}$, so we only conclude that the Rauzy–Veech group has index at most 28.

If the genus is greater than three, we get sequences of simple extensions starting from both nonhyperelliptic components of minimal Abelian strata, so the Rauzy–Veech group is equal to its entire ambient symplectic group.

For genus two, only the strata $\mathbb{Q}(6,-1,-1)$ and $\mathbb{Q}(3,3,-1,-1)$ are not connected. The generalised permutations below represent their nonhyperelliptic components. Moreover, erasing the letters A and B produces representatives of $\mathcal{H}(2)$ and $\mathcal{H}(1,1)$, respectively:

$$\begin{pmatrix} 1 & 2 & 3 & A & A & 4 \\ 4 & 3 & B & B & 2 & 1 \end{pmatrix} \quad \text{and} \quad \begin{pmatrix} 1 & 2 & A & A & 3 & 4 & 5 \\ 5 & B & B & 4 & 3 & 2 & 1 \end{pmatrix}.$$

The index of the Rauzy–Veech group of $\mathcal{H}(2)$ is 6, while the Rauzy–Veech group of $\mathcal{H}(1,1)$ is equal to its entire ambient symplectic group.

3.5.4 Exceptional strata

There exist four exceptional strata that satisfy our hypothesis: $\mathbb{Q}(6,3,-1)$ and $\mathbb{Q}(3,3,3,-1)$ in genus 3 and $\mathbb{Q}(6,3,3)$ and $\mathbb{Q}(3,3,3,3)$ in genus 4. These strata have two nonhyperelliptic connected components, usually called *regular* and *irregular* [Lan08; CM14]. For these cases, we can find explicit simple extensions, shown in Table 3.1, to prove that their Rauzy–Veech groups are equal to their entire ambient symplectic groups. They start from either a connected component of the moduli space of Abelian differentials whose Rauzy–Veech group is its entire symplectic subgroup, or from two different connected components of minimal Abelian strata.

These simple extensions were found by using the `surface_dynamics` package for SageMath to obtain explicit representatives for the exceptional connected components. We then performed a depth-first scan on their Rauzy classes.

Start	End	Generalised permutation
$\mathcal{H}(4)^{\text{hyp}}$	$\mathbb{Q}(6,3,-1)^{\text{reg}}$	$\begin{pmatrix} 1 & 2 & 3 & A & 4 & A & 5 & 6 \\ 6 & 5 & 4 & 3 & 2 & B & B & 1 \end{pmatrix}$

$\mathcal{H}(4)^{\text{odd}}$	$\mathcal{Q}(6,3,-1)^{\text{reg}}$	$\begin{pmatrix} 1 & 2 & 3 & 4 & A & 5 & A & 6 \\ 6 & 4 & B & B & 2 & 5 & 3 & 1 \end{pmatrix}$
$\mathcal{H}(4)^{\text{hyp}}$	$\mathcal{Q}(6,3,-1)^{\text{irr}}$	$\begin{pmatrix} 1 & A & 2 & 3 & 4 & 5 & A & 6 \\ 6 & B & B & 5 & 4 & 3 & 2 & 1 \end{pmatrix}$
$\mathcal{H}(4)^{\text{odd}}$	$\mathcal{Q}(6,3,-1)^{\text{irr}}$	$\begin{pmatrix} 1 & 2 & 3 & 4 & 5 & A & A & 6 \\ 6 & B & 3 & B & 5 & 2 & 4 & 1 \end{pmatrix}$
$\mathcal{H}(3,1)$	$\mathcal{Q}(3,3,3,-1)^{\text{reg}}$	$\begin{pmatrix} 1 & A & A & 2 & 3 & 4 & 5 & 6 & 7 \\ 7 & 6 & B & 5 & 2 & B & 4 & 3 & 1 \end{pmatrix}$
$\mathcal{H}(3,1)$	$\mathcal{Q}(3,3,3,-1)^{\text{irr}}$	$\begin{pmatrix} 1 & 2 & 3 & 4 & 5 & A & 6 & A & 7 \\ 7 & 6 & 2 & B & B & 5 & 4 & 3 & 1 \end{pmatrix}$
$\mathcal{H}(6)^{\text{even}}$	$\mathcal{Q}(6,3,3)^{\text{reg}}$	$\begin{pmatrix} 1 & 2 & A & 3 & 4 & 5 & 6 & 7 & A & 8 \\ 8 & 7 & 5 & B & 2 & 6 & B & 4 & 3 & 1 \end{pmatrix}$
$\mathcal{H}(6)^{\text{odd}}$	$\mathcal{Q}(6,3,3)^{\text{reg}}$	$\begin{pmatrix} 1 & A & 2 & 3 & 4 & 5 & A & 6 & 7 & 8 \\ 8 & 4 & 7 & B & 5 & 3 & 6 & B & 2 & 1 \end{pmatrix}$
$\mathcal{H}(6)^{\text{even}}$	$\mathcal{Q}(6,3,3)^{\text{irr}}$	$\begin{pmatrix} 1 & 2 & 3 & 4 & A & 5 & A & 6 & 7 & 8 \\ 8 & 7 & 5 & B & 2 & 6 & B & 4 & 3 & 1 \end{pmatrix}$
$\mathcal{H}(6)^{\text{odd}}$	$\mathcal{Q}(6,3,3)^{\text{irr}}$	$\begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 & A & 7 & A & 8 \\ 8 & B & 5 & B & 3 & 7 & 4 & 6 & 2 & 1 \end{pmatrix}$
$\mathcal{H}(3,3)^{\text{nonhyp}}$	$\mathcal{Q}(3,3,3,3)^{\text{reg}}$	$\begin{pmatrix} 1 & A & 2 & A & 3 & 4 & 5 & 6 & 7 & 8 & 9 \\ 9 & 6 & B & 5 & 3 & 7 & 2 & 8 & B & 4 & 1 \end{pmatrix}$
$\mathcal{H}(3,3)^{\text{nonhyp}}$	$\mathcal{Q}(3,3,3,3)^{\text{irr}}$	$\begin{pmatrix} 1 & A & 2 & A & 3 & 4 & 5 & 6 & 7 & 8 & 9 \\ 9 & 5 & 2 & 6 & 4 & 3 & B & 8 & B & 7 & 1 \end{pmatrix}$

Table 3.1: Table of extensions for nonhyperelliptic connected components of exceptional strata. The generalised permutation in the third column belongs to the connected component in the second column. Erasing letters A and B produces a permutation belonging to the Abelian connected component in the first column.

3.6 Simplicity of the Lyapunov spectra

In this section, we will show that if a fixed stratum $\mathcal{S}$ satisfies the hypothesis of Theorem 3.1.1, then its ("plus" or "minus", depending on the stratum) Lyapunov spectrum is simple. Fix a generalised permutation π representing $\mathcal{S}$ and constructed by a sequence of simple extensions starting from a minimal Abelian stratum, as in the proof of Theorem 3.1.1.

We have defined the Rauzy–Veech group of a stratum by allowing the arrows of a Rauzy class to be reversed. The next lemma shows that this coincides with the group generated by *directed* cycles and their inverses.

Lemma 3.6.1. *The group generated by the ("plus" or "minus") Rauzy–Veech monoid of $\mathcal{S}$ is equal to* *the ("plus" or "minus", respectively) Rauzy–Veech group of $\mathcal{S}$.*

Proof. The proof is purely combinatorial and it is therefore identical for the "plus" and "minus" cases. It is essentially a consequence of the fact that Rauzy classes are strongly connected.

Let $\mathcal{R}$ be the Rauzy class of π and let $\widetilde{\mathcal{R}}$ be an undirected copy. Let $\gamma = \gamma_1^{\varepsilon_1}\gamma_2^{\varepsilon_2}\cdots\gamma_n^{\varepsilon_n}$ be a cycle in $\widetilde{\mathcal{R}}$, where each γ_i is an arrow in $\mathcal{R}$ and the $\varepsilon_i \in \{-1, +1\}$ are used to denote either an arrow of $\mathcal{R}$ or its reversed copy. Let $0 = i_0 < i_1 < i_2 < \cdots < i_k = n$ be indices such that $\varepsilon_{i_j+1} = \varepsilon_{i_j+2} = \cdots = \varepsilon_{i_{j+1}}$ and $\varepsilon_{i_j} \neq \varepsilon_{i_{j+1}}$ for each $0 \leq j < k$. Assume that $\varepsilon_1 = +1$, since the other case is similar.

We define the following oriented cycles:

$$c_1 = \gamma_1\gamma_2\cdots\gamma_{i_1}w_1$$

where w_1 is any path in $\mathcal{R}$ joining the end of γ_{i_1} with π;

$$c_2 = w_2\gamma_{i_2}\gamma_{i_2-1}\cdots\gamma_{i_1+1}w_1$$

where w_2 is any path in $\mathcal{R}$ joining π with the start of γ_{i_2};

$$c_3 = w_2\gamma_{i_2+1}\gamma_{i_2+2}\cdots\gamma_{i_3}w_3$$

where w_3 is any path in $\mathcal{R}$ joining the end of γ_{i_3} with π; and continue in this way inductively. We choose the last w_j as the zero-length path joining π and π. Then, the matrix B_γ is equal to $\cdots B_{c_3}B_{c_2}^{-1}B_{c_1}$ as the matrices B_{w_j} cancel out. We conclude that B_γ belongs to the group generated by the Rauzy–Veech monoid of π, so it contains the Rauzy–Veech group of π. □

We obtain that the Rauzy–Veech monoid of $\mathcal{S}$ is Zariski-dense. By the work of Benoist [Ben97], it is also pinching and twisting.

What remains to obtain the simplicity of the Lyapunov spectrum follows from standard arguments using the general criterion by Avila and Viana [AV07b]. We will therefore sketch the steps without too many details. First, it is important to consider that not every element of the Rauzy–Veech monoid represents an orbit of the Teichmüller geodesic flow. A standard way to ensure that a Teichmüller orbit follows a given Rauzy–Veech orbit is using some "completeness" condition. More precisely, we say that a walk γ in the Rauzy class of π is *k-complete* if every letter of $\mathcal{A}$ wins at least k times in γ. It is clear that k-complete walks can be found, as Rauzy classes are strongly connected. Now, let γ^* be a k-complete cycle γ^* at π such that if $\gamma^* = \gamma_s\gamma = \gamma\gamma_e$ then γ is either γ^* or trivial. By the work of Avila and Resende [AR12, Section 6.2], if k is sufficiently large then any cycle γ satisfying the following three conditions produces an orbit of the Teichmüller geodesic flow:

- γ starts with γ^*;
- γ ends with γ^*;

- γ does not start with $\gamma^*\gamma^*$.

We write $\gamma = \gamma^* w \gamma^*$. To conclude, we observe that monoid induced by these cycles is also Zariski-dense, as the entries of B_γ depend polynomially on the entries of B_w. This shows that this monoid is pinching and twisting, and, therefore, that the Lyapunov spectrum of the Teichmüller geodesic flow on $\mathcal{S}$ is simple, so it concludes the proof.

Chapter 4

Realisability of some quaternionic monodromy groups

The purpose of this chapter is to show that some groups of the form $\mathrm{SO}^*(2d)$ are realisable as monodromy groups of square-tiled surfaces. It is an adapted version of the article "A family of quaternionic monodomy groups of the Kontsevich–Zorich cocycle" [Gut19a].

4.1 Introduction

Let $\mathcal{M}$ be an affine invariant submanifold. Recall from Chapter 1 that the possible Zariski-closures of the monodromy groups arising from $\mathrm{SL}(2, \mathbb{R})$-(strongly-)irreducible subbundles, at the level of real Lie algebra representations and up to compact factors, belong to the following list:

(i) $\mathfrak{sp}(2g, \mathbb{R})$ in the standard representation;
(ii) $\mathfrak{su}(p, q)$ in the standard representation;
(iii) $\mathfrak{su}(p, 1)$ in an exterior power representation;
(iv) $\mathfrak{so}^*(2d)$ in the standard representation; or
(v) $\mathfrak{so}_{\mathbb{R}}(n, 2)$ in a spin representation.

Nevertheless, it is not known whether every Lie algebra representation in this list is realisable as a monodromy group [Fil17, Question 1.5]. Indeed, it is well-known that every group in the first item is realisable. The groups in the second item were shown to be realisable by Avila, Matheus and Yoccoz [AMY19]. Moreover, the group $\mathrm{SO}^*(6)$ in its standard representation (which coincides with $\mathrm{SU}(3, 1)$ in its second exterior power representation) is also realisable by the work of Filip, Forni and Matheus [FFM18].

The main theorem of this chapter is the following:

Theorem 4.1.1. *For each d belonging to a density-$1/8$ subset of the natural numbers, there exists a square-tiled surface conjecturally realising the group $\mathrm{SO}^*(2d)$ as the monodromy group of an $\mathrm{SL}(2, \mathbb{R})$-(strongly-)irreducible piece of its Kontsevich–Zorich cocycle. This conjecture depends on certain linear-*

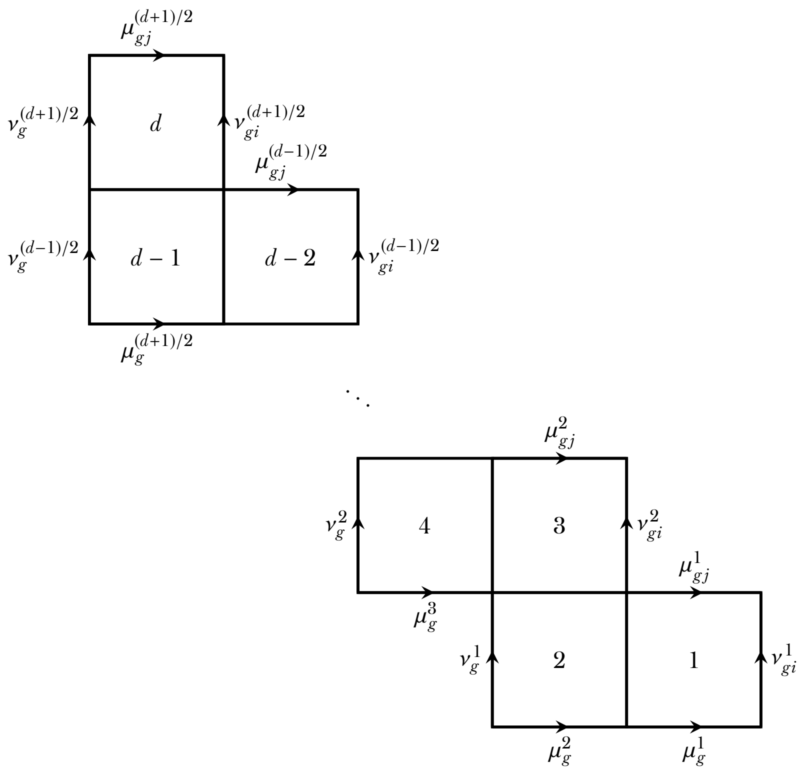


Figure 4.1: An illustration of $M_g^{(d)}$.

algebraic conditions, which can be computationally shown to be true for small values of d. In this way, we show that $\mathrm{SO}^*(2d)$ *is realisable for every* $11 \leq d \leq 299$ *in the congruence class* $d = 3 \bmod 8$*, except possibly for* $d = 35$ *and* $d = 203$*.*

Indeed, as was done by Filip, Forni and Matheus [FFM18], we will show that these groups seem to arise in quaternionic covers of simple square-tiled surfaces.

4.2 Construction of the family of square-tiled surfaces

In this section, we will construct the quaternionic covers that realise the desired groups as the Zariski-closure of the monodromy of a specific flat irreducible subbundle of the Hodge bundle.

Let $d \geq 3$ be an odd integer. We consider a "staircase" $M^{(d)}$ with d squares: the square-tiled surface induced by the horizontal permutation $(2,1)(4,3)\dots(d-1,d-2)(d)$ and the vertical permutation $(1)(2,3)(4,5)\cdots(d-1,d)$. It belongs to the component $\mathcal{H}_{(d+1)/2}(d-1)^{\mathrm{hyp}}$. Its automorphism group is trivial, since it belongs to a minimal stratum.

We construct a cover $\widetilde{M}^{(d)}$ of $M^{(d)}$ as follows: for each element g of the quaternion group $Q = \{1, -1, i, -i, j, -j, k, -k\}$, we take a copy $M_g^{(d)}$ of $M^{(d)}$. We glue the r-th right vertical side of $M_g^{(d)}$ to the r-th left vertical side of $M_{gi}^{(d)}$. Similarly, we glue the r-th top horizontal side of $M_g^{(d)}$ to the r-th bottom horizontal side of $M_{gj}^{(d)}$. See Figure 4.1. This construction coincides,

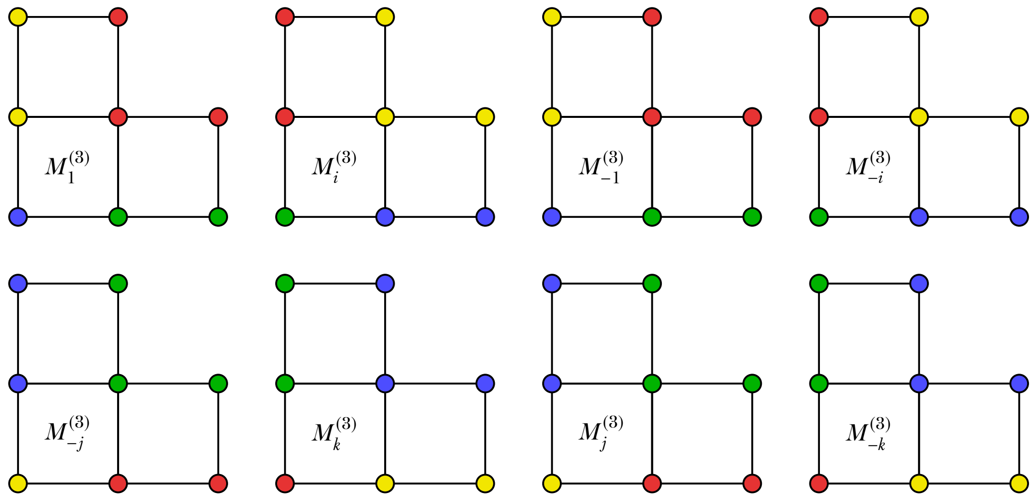


Figure 4.2: An illustration of $\widetilde{M}^{(3)}$ showing its four singularities. Horizontally, the each copy of $M^{(3)}$ is cyclically glued to the copy on its right or left, but this does not hold for the vertical gluings (as the top sides of $M^{(3)}_k$, for example, are glued to the bottom sides of $M^{(3)}_{-i}$).

up to relabelling, with that of Filip, Forni and Matheus for $d = 3$ [FFM18, Section 5.1].

For each $g \in Q$, we can define an automorphism φ_g of $\widetilde{M}^{(d)}$ by mapping $M^{(d)}_h$ to $M^{(d)}_{gh}$ in the natural way, that is, preserving the covering map $\widetilde{M}^{(d)} \to M^{(d)}$ for each $h \in Q$. Indeed, the gluings are defined by multiplication on the right, which commutes with multiplication on the left. These are the only automorphisms of $\widetilde{M}^{(d)}$: an automorphism ψ of $\widetilde{M}^{(d)}$ induces an automorphism of $M^{(d)}$ by "forgetting the labels". Since the only automorphism of $M^{(d)}$ is the identity, $M^{(d)}_1$ is mapped to some $M^{(d)}_g$ for $g \in Q$ in a way that preserves the covering map $\widetilde{M}^{(d)} \to M^{(d)}$. Thus, $\psi = \varphi_g$ and $\mathrm{Aut}(\widetilde{M}^{(d)}) \simeq Q$. We will denote $\mathrm{Aut}(\widetilde{M}^{(d)})$ by G.

From now on, we will restrict to the case $d = 3 \bmod 8$. The surface $\widetilde{M}^{(d)}$ has four singularities, each of order $2d-1$. Therefore, $\widetilde{M}^{(d)}$ belongs to the (connected) stratum $\mathcal{H}_{4d-1}((2d-1)^4)$.

Since the automorphism $\varphi_{-1} \in G$ is an involution, it induces a splitting

$$H_1(\widetilde{M}^{(d)}; \mathbb{R}) = H_1^+(\widetilde{M}^{(d)}) \oplus H_1^-(\widetilde{M}^{(d)}),$$

where $H_1^\pm(\widetilde{M}^{(d)})$ is the subspace of $H_1(\widetilde{M}^{(d)})$ where φ_{-1} acts as $\pm\mathrm{Id}$. These subspaces are symplectic and symplectically orthogonal. The subspace $H_1^+(\widetilde{M}^{(d)})$ contains $H_1^{\mathrm{st}}(\widetilde{M}^{(d)})$ and is naturally isomorphic to $H_1(M^{(d)}_\pm; \mathbb{R})$, where $M^{(d)}_\pm = \widetilde{M}^{(d)}/\varphi_{-1}$. This latter surface is an intermediate cover of $M^{(d)}$ over the group $Q/\{1,-1\} \simeq \mathbb{Z}/2\mathbb{Z} \times \mathbb{Z}/2\mathbb{Z}$. Since every singularity of $\widetilde{M}^{(d)}$ is fixed by φ_{-1}, $M^{(d)}_\pm$ belongs to the stratum $\mathcal{H}_{2d-1}((d-1)^4)$. Therefore, $H_1^+(\widetilde{M}^{(d)})$ is a $(4d-2)$-dimensional subspace of the $(8d-2)$-dimensional space $H_1(\widetilde{M}^{(d)}; \mathbb{R})$ and we obtain that the dimension of $H_1^-(\widetilde{M}^{(d)})$ is $4d$.

The irreducible representations (over $\mathbb{C}$) of the group Q can be summarised in the following character table:

As detailed in Section 1.2.4, $H_1(\widetilde{M}^{(d)}; \mathbb{R})$ can be split into isotypical components associated

	Dimension	1	-1	$\pm i$	$\pm j$	$\pm k$
χ_1	1	1	1	1	1	1
χ_i	1	1	1	1	-1	-1
χ_j	1	1	1	-1	1	-1
χ_k	1	1	1	-1	-1	1
tr χ_2	2	2	-2	0	0	0

Table 4.1: Character table of Q.

with such representations. From the character table, we obtain that $H_1^-(\widetilde{M}^{(d)})$ corresponds to $2d$ copies of a G-irreducible representation whose character is the quaternionic character χ_2, that is, $H_1^-(\widetilde{M}^{(d)}) = W_{\chi_2}$. Indeed, φ_{-1} acts as the identity for any other representation in the table. We obtain the following:

Lemma 4.2.1. *The Zariski-closure of the monodromy group of the flat subbundle induced by $H_1^-(\widetilde{M}^{(d)})$ is a subgroup of $\mathrm{SO}^*(2d)$. Moreover, Kontsevich–Zorich cocycle over this subbundle has at least four zero Lyapunov exponents.*

Proof. The first statement is a direct consequence of Theorem 1.2.20. The second statement is a consequence of the first since d is odd [Fil17, Corollary 5.5, Section 5.3.4]. □

We will prove that, for certain d with $d \bmod 8 = 3$, such Zariski-closure is actually $\mathrm{SO}^*(2d)$.

4.3 Computation of the monodromy groups

4.3.1 Dimensional constraints

In the presence of zero Lyapunov exponents, Theorem 1.2.14 states that the only possible Lie algebra representations of the Zariski-closure of the monodromy group of a flat subbundle are $\mathfrak{so}^*(2d)$ in the standard representation $\rho_{2d} \colon \mathfrak{so}^*(2d) \to \mathbb{R}^{4d}$, $\mathfrak{su}(p, q)$ in the standard representation $\sigma_{p,q} \colon \mathfrak{su}^*(p, q) \to \mathbb{R}^{2(p+q)}$, and $\mathfrak{su}(p, 1)$ in some exterior power representation $\tau_{r,p} \colon \mathfrak{su}(p, 1) \to \mathbb{R}^{2\binom{p+1}{r}}$. The representations $\tau_{r,p}$ are irreducible and faithful: by complexifying, one obtains $\mathfrak{sl}(p + 1, \mathbb{C})$ whose exterior power representations are known to satisfy these properties.

Let

$$\mathcal{D} = \left\{ d \in \mathbb{N} \;\middle|\; 2d \neq \binom{p+1}{r} \text{ for every } p \text{ and } 1 < r < p \right\}.$$

We have the following dimensional constraints:

Lemma 4.3.1. *Let $d \in \mathcal{D}$. If $2d = p + q$, then $\dim_{\mathbb{R}} \mathfrak{so}^*(2d) < \dim_{\mathbb{R}} \mathfrak{su}(p, q)$. Moreover, if $2d = \binom{p+1}{r}$, then $\dim_{\mathbb{R}} \mathfrak{so}^*(2d) < \dim_{\mathbb{R}} \mathfrak{su}(p, 1)$.*

Proof. We have that

$$\dim_{\mathbb{R}} \mathfrak{so}^*(2d) = d(2d - 1)$$

$$\dim_{\mathbb{R}} \mathfrak{su}(p, q) = (p + q)^2 - 1$$
$$\dim_{\mathbb{R}} \mathfrak{su}(p, 1) = p(p + 2).$$

If $p + q = 2d$, then $\dim_{\mathbb{R}} \mathfrak{su}(p, q) = 4d^2 - 1$. Thus, $\dim_{\mathbb{R}} \mathfrak{so}^*(2d) < \dim_{\mathbb{R}} \mathfrak{su}(p, q)$ for every d. If $2d = \binom{p+1}{r}$ for $d \in \mathcal{D}$, then we have that $r \in \{1, p\}$. Therefore, $2d = p + 1$ and we conclude as in the previous case. □

Remark 4.3.2. To obtain the strict inequality in the previous proof, it is necessary to assume that $d \in \mathcal{D}$. Indeed, if $d \notin \mathcal{D}$ then $2d = \binom{p+1}{r}$ with $1 < r < p$. This is enough to show that $\dim_{\mathbb{R}} \mathfrak{su}(p, 1) \leq \dim_{\mathbb{R}} \mathfrak{so}^*(2d)$. Indeed, since $2d = \binom{p+1}{r} \geq \binom{p+1}{2}$ we have that $p(p + 1) \leq 4d$ and it is easy to check that this results in $p(p + 2) \leq d(2d - 1)$ if $d \geq 3$.

The previous lemma shows that, assuming (strong) irreducibility, $\mathfrak{so}^*(2d)$ is the only possible Lie algebra of the Zariski-closure of the flat subbundle induced by $H_1(\widetilde{X}^{(d)})$. Indeed, we already know by Lemma 4.2.1 that its Lie algebra $\mathfrak{g}$ is contained in $\mathfrak{so}^*(2d)$, so clearly $\dim_{\mathbb{R}} \mathfrak{g} \leq \dim_{\mathbb{R}} \mathfrak{so}^*(2d)$. Moreover, the corresponding Lie algebra representation is either ρ_{2d}, $\sigma_{p,q}$ or $\tau_{r,p}$. These representations act irreducibly on real vector spaces of dimensions $4d$, $2(p+q)$ and $2\binom{p+1}{r}$, respectively. Hence, if the sought representation is $\sigma_{p,q}$ or $\tau_{r,p}$, the previous lemma implies that $\dim_{\mathbb{R}} \mathfrak{so}^*(2d) < \dim_{\mathbb{R}} \mathfrak{g}$, a contradiction.

We finish this section by showing that the set $\mathcal{D}$ is large inside $\mathbb{N}$:

Lemma 4.3.3. *The set $\mathcal{D}$ has full density in $\mathbb{N}$.*

Proof. Let
$$B_{p+1} = \left\{ \binom{p+1}{r} \;\middle|\; 1 < r < p \right\} \text{ and } B = \bigcup_{p \geq 3} B_{p+1}.$$
We will show that $|B \cap \{1, \ldots, n\}|/n \to 0$. Observe that
$$|B \cap \{1, \ldots, n\}| \leq \sum_{p \geq 3} |B_{p+1} \cap \{1, \ldots, n\}|$$
Now, observe that:

- If $p \geq 3$ and $\binom{p+1}{2} > n$, then $|B_{p+1} \cap \{1, \ldots, n\}| = 0$;
- If $p \geq 5$ and $\binom{p+1}{4} > n$, then $|B_{p+1} \cap \{1, \ldots, n\}| \leq 2$.

We will split the sum in this way to obtain a bound for $|B \cap \{1, \ldots, n\}|$. Let p_2 be the smallest $p \geq 3$ such that $\binom{p+1}{2} > n$ and let p_4 be the smallest $p \geq 5$ such that $\binom{p+1}{4} > n$. We have that
$$\begin{aligned} |B \cap \{1, \ldots, n\}| &\leq \sum_{p=3}^{p_4 - 1} (p + 1) + \sum_{p=p_4}^{p_2 - 1} 2 \leq p_4(p_4 - 1) + 2(p_2 - 1) \\ &= O(n^{1/4})O(n^{1/4}) + O(n^{1/2}) = O(n^{1/2}) = o(n). \end{aligned}$$
□

4.3.2 Dehn multi twists

We will use Dehn multi twists along specific rational directions to prove irreducibility. Assume that there exist rational directions (p_r, q_r) for $0 \le r < d$ such that:

1. the cylinder decomposition along (p_r, q_r) consists of eight cylinders with waist curves c_g^r, for $g \in Q$, of the same length. Thus, the Dehn multi twist along (p_r, q_r) can be written as $T_r v = v + n_r \sum_{g\in G} \langle v, c_g^r\rangle c_g^r$; and
2. the action of G on the labels is "well-behaved", that is, $(\varphi_h)_* c_g^r = c_{hg}^r$ for every $0 \le r < d$, and $g, h \in Q$.

Let $Q^+ = \{1, i, j, k\}$ and $\hat{c}_g^r = c_g^r - c_{-g}^r$ for each $g \in Q^+$. If $v \in H_1^-(\widetilde{M}^{(d)})$ we have that

$$\begin{aligned}
\langle v, \hat{c}_g^r\rangle \hat{c}_g^r &= \langle v, c_g^r - c_{-g}^r\rangle (c_g^r - c_{-g}^r) \\
&= \langle v, c_g^r\rangle c_g^r - \langle v, c_{-g}^r\rangle c_g^r - \langle v, c_g^r\rangle c_{-g}^r + \langle v, c_{-g}^r\rangle c_{-g}^r \\
&= \langle v, c_g^r\rangle c_g^r + \langle v, c_g^r\rangle c_g^r + \langle v, c_{-g}^r\rangle c_{-g}^r + \langle v, c_{-g}^r\rangle c_{-g}^r \\
&= 2(\langle v, c_g^r\rangle c_g^r + \langle v, c_{-g}^r\rangle c_{-g}^r),
\end{aligned}$$

where we used that $(\varphi_{-1})_* v = -v$ and that $(\varphi_{-1})_*$ is a symplectic automorphism. Therefore, $T_r v = v + \frac{n_r}{2} \sum_{g\in Q^+} \langle v, \hat{c}_g^r\rangle \hat{c}_g^r$. Let $C_r = \langle \hat{c}_g^r\rangle_{g\in Q^+}$, the span of the $\hat{c}_g^r$ for $g \in Q^+$. We will also assume the following:

3. $\{\hat{c}_g^r\}_{g,r}$ is a basis of $H_1^-(\widetilde{M}^{(d)})$;
4. for each $0 \le r \ne s < d$ and $v \in C_r \setminus \{0\}$, $T_s v \ne v$; and
5. for any $v \in C_0 \setminus \{0\}$, $C_0 = \langle \{v\} \cup \{(T_0 - \mathrm{Id})(T_r - \mathrm{Id})(T_1 - \mathrm{Id})v\}_{r=2}^4\rangle$.

The last three conditions can be stated in terms of intersection numbers:

3. the matrix of intersection numbers of the $\hat{c}_g^r$ is nonsingular;
4. for each $0 \le r \ne s < d$, there exists $g \in Q^+$ such that $\langle \hat{c}_g^r, \hat{c}_0^s\rangle \ne 0$; and
5. let $v = \sum_{g\in Q^+} \mu_g \hat{c}_g^0 \in C_0 \setminus \{0\}$ and $v_r = (T_0 - \mathrm{Id})(T_r - \mathrm{Id})(T_1 - \mathrm{Id})v$ for $2 \le r \le 4$. Write $v_r = \sum_{g\in Q^+} \mu_{g,r} \hat{c}_g^0$ and put $\mu_{g,1} = \mu_g$. Then, the matrix of coefficients $(\mu_{g,r})_{g\in Q^+, 1\le r\le 4}$ is nonsingular.

These conditions are enough to prove that $H_1^-(\widetilde{M}^{(d)})$ is strongly irreducible for the action of $\mathrm{Aff}_*(\widetilde{M}^{(d)})$:

Lemma 4.3.4. *Assume that* (1)–(5) *hold. Then,* $\mathrm{Aff}_{**}(\widetilde{M}^{(d)})$ *acts irreducibly on* $H_1^-(\widetilde{M}^{(d)})$*, where* $\mathrm{Aff}_{**}(\widetilde{M}^{(d)})$ *is any finite-index subgroup of* $\mathrm{Aff}_*(\widetilde{M}^{(d)})$.

Proof. Let $V \ne \{0\}$ be a subspace of $H_1^-(\widetilde{M}^{(d)})$ on which $\mathrm{Aff}_{**}(\widetilde{M}^{(d)})$ acts irreducibly. By (3), it is enough to prove that $C_r \subseteq V$ for each $0 \le r < d$.

Since the index of $\mathrm{Aff}_{**}(\widetilde{M}^{(d)})$ is finite, some power of T_r belongs to $\mathrm{Aff}_{**}(\widetilde{M}^{(d)})$ for every $0 \le r < d$. Without loss of generality, we can assume $T_r \in \mathrm{Aff}_{**}(\widetilde{M}^{(d)})$, since the number $n_r \ne 0$ in the formula for T_r is irrelevant for the proof.

We will first show that $C_0 \subseteq V$. Let $u \in V \setminus \{0\}$. Since $H_1^-(\widetilde{M}^{(d)})$ is symplectic, by (3) there exists $0 \le r < d$ such that $T_r u \ne u$. Clearly, $w = (T_r - \mathrm{Id})u \in C_r \setminus \{0\}$. Now, by (4),

$v = (T_0 - \mathrm{Id})w \in C_0 \setminus \{0\}$. Finally, by (5) we have that $C_0 \subseteq V$.

Now, it is enough to show that $(T_r - \mathrm{Id})C_0 = C_r$. Indeed, this implies that $C_r \subseteq V$ for each $0 \leq r < d$ as $C_0 \subseteq V$ and V is closed under addition and is irreducible for the action of $\mathrm{Aff}_{**}(\widetilde{X}^{(d)})$. Let $v = (T_r - \mathrm{Id})\hat{c}_1^0 \in C_r \setminus \{0\}$. Observe that (2) implies that C_0 is G-invariant. Since $\mathrm{Aff}_{**}(\widetilde{M}^{(d)})$ commutes with G, we have that $(\varphi_g)_* v \in V$ for each $g \in Q^+$. Write $v = \sum_{g \in Q^+} \mu_g \hat{c}_g^r$. By (2), we have that

$$\begin{aligned}
(\varphi_i)_* v &= -\mu_i \hat{c}_1^r + \mu_1 \hat{c}_i^r - \mu_k \hat{c}_j^r + \mu_j \hat{c}_k^r \\
(\varphi_j)_* v &= -\mu_j \hat{c}_1^r + \mu_k \hat{c}_i^r + \mu_1 \hat{c}_j^r - \mu_i \hat{c}_k^r \\
(\varphi_k)_* v &= -\mu_k \hat{c}_1^r - \mu_j \hat{c}_i^r + \mu_i \hat{c}_j^r + \mu_1 \hat{c}_k^r .
\end{aligned}$$

Therefore, the matrix of coefficients of $(\varphi_g)_* v$ for $g \in Q^+$ is

$$\begin{pmatrix}
\mu_1 & \mu_i & \mu_j & \mu_k \\
-\mu_i & \mu_1 & -\mu_k & \mu_j \\
-\mu_j & \mu_k & \mu_1 & -\mu_i \\
-\mu_k & -\mu_j & \mu_i & \mu_1
\end{pmatrix}$$

whose determinant is $\left(\sum_{g \in Q^+} \mu_g^2\right)^2 \neq 0$. We obtain that $\langle (\varphi_g)_* v \rangle_{g \in Q^+} = C_r \subseteq V$, which completes the proof. □

Remark 4.3.5. The action of the automorphism group $G \simeq Q$ on $H_1^-(\widetilde{M}^{(d)})$ induces a structure of an $\mathbb{H}$-module on $H_1^-(\widetilde{M}^{(d)})$. Indeed, the map $(\varphi_g)_*$ can be interpreted as the multiplication by g. The blocks C_r satisfy $C_r = \langle (\varphi_g)_* v \rangle_{g \in Q^+}$, as shown in the proof of the previous lemma, so they can be interpreted as the span of single vectors by the coefficients in $\mathbb{H}$. In this context, conditions (1)–(4) are a "moral analogue" of Deligne's criterion to compute Zariski-closures [Del80; PS03]: the group contains appropriate transvections along a basis of the vector space and this basis is transformed by the transvections in a sufficiently rich manner. These conditions are indeed enough to prove the strong irreducibility of $H_1^-(\widetilde{M}^{(d)})$ in the quaternionic setting, that is, by the action of the group generated by $\mathrm{Aff}_{**}(\widetilde{M}^{(d)})$ and G. Nevertheless, since we are interested in the *real* Zariski-closure as opposed to the *quaternionic* Zariski-closure of the monodromy group, we also need condition (5) to ensure that any invariant subspace contains an entire quaternionic block C_0. Of course, the block C_0 could be replaced by any other block, but we stated condition (5) in this way for simplicity.

We can now show that this conditions are enough for the monodromy group to be $\mathrm{SO}^*(2d)$:

Proposition 4.3.6. *Assume that* $d = 3 \bmod 8$*, that* $d \in \mathcal{D}$ *and that* (1)–(5) *hold. Then, the Zariski-closure of the group* $\rho(\mathrm{Aff}_*(M))|_{H_1^-(\widetilde{M}^{(d)})}$ *is* $\mathrm{SO}^*(2d)$.

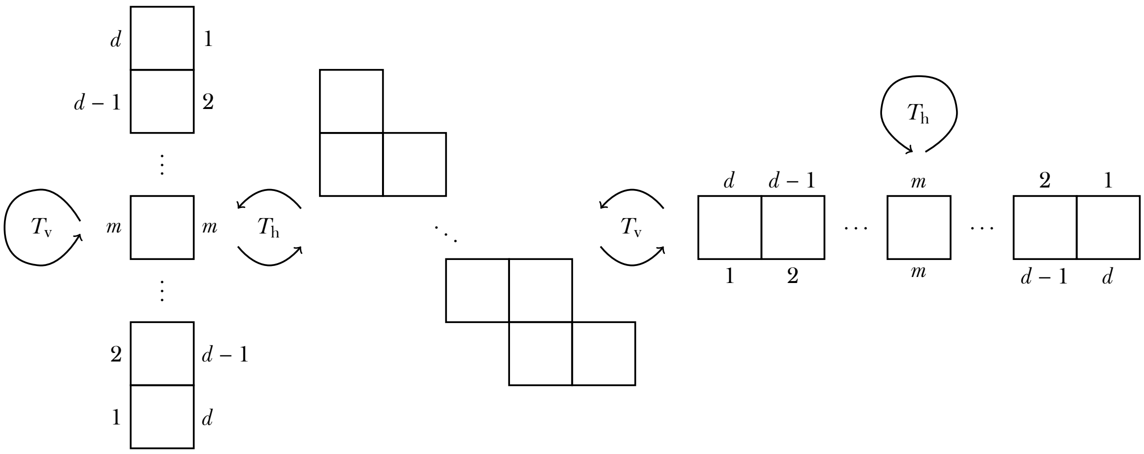


Figure 4.3: The $\mathrm{SL}(2,\mathbb{Z})$-orbit of $M^{(d)}$ using T_h and T_v as generators. It consists of three distinct square-tiled surfaces, which we call $O^{(d)}$, $M^{(d)}$ and $N^{(d)}$ from left to right. The labels in the $N^{(d)}$ and $O^{(d)}$ show the identification of the sides. Unlabelled horizontal sides are identified with the only horizontal having the same horizontal coordinates, and similarly for unlabelled vertical sides.

Proof. By Lemma 4.2.1, exactly four Lyapunov exponents of the Kontsevich–Zorich cocycle are zero, so the hypotheses of Theorem 1.2.14 are satisfied. To conclude by Lemma 4.3.1, it is enough for $\mathrm{Aff}_*(M)$ to act strongly irreducibly on $H_1^-(\widetilde{M}^{(d)})$, which follows from the previous lemma. □

The next section is then devoted to finding the desired Dehn multi twists.

4.3.3 Suitable rational directions

In this section, we will find the desired rational directions (p_r, q_r) and prove (1)–(5) for the specific values of d mentioned in the statement of the main theorem to conclude the proof. Assume that $d = 3 \bmod 8$ for the rest of the section.

Recall from Chapter 1 that the matrices

$$T_\mathrm{h} = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}, \quad T_\mathrm{v} = \begin{pmatrix} 1 & 0 \\ 1 & 1 \end{pmatrix}$$

generate $\mathrm{SL}(2,\mathbb{Z})$ and, thus, can be used to understand the $\mathrm{SL}(2,\mathbb{Z})$-orbit of a square-tiled surface. The orbit of the "staircase" $M^{(d)}$ consists of three elements, which we call $O^{(d)}$, $M^{(d)}$ and $N^{(d)}$. See Figure 4.3. Its Veech group has index three and is the so-called theta subgroup of $\mathrm{SL}(2,\mathbb{Z})$ consisting on the elements whose modulo-two reductions are either Id or $\left(\begin{smallmatrix} 0 & 1 \\ 1 & 0 \end{smallmatrix}\right)$. See Figure 4.4.

Since $\widetilde{M}^{(d)}$ is a cover of $M^{(d)}$, the graph induced by the action on T_h and T_v on $\widetilde{M}^{(d)}$ is a cover of the graph in Figure 4.3. In other words, if $g \in \mathrm{SL}(2,\mathbb{Z})$ then $g \cdot \widetilde{M}^{(d)}$ is a degree-eight cover of $g \cdot M^{(d)}$. Moreover, since the graph in Figure 4.3 has only three vertices, writing g

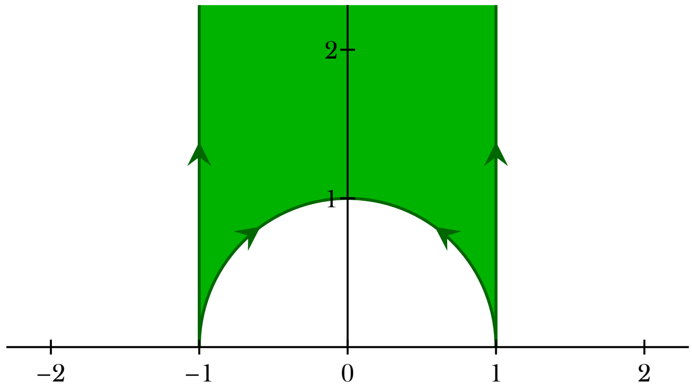


Figure 4.4: The “canonical” fundamental domain of the action of the theta subgroup on the upper half-plane. The resulting Teichmüller curve has genus zero and two cusps.

d	Index	Genus	Cusps
3	12	0	3
11	16896	225	960
19	1867776	30721	94208

Table 4.2: The index of $\mathrm{SL}(\widetilde{M}^{(d)})$ and the genus and number of cusps of the resulting Teichmüller curve for small values of d.

in terms of T_{h} and T_{v} and following the arrows of the graph us to compute $g \cdot M^{(d)}$, which is useful to understand $g \cdot \widetilde{M}^{(d)}$. The index of $\mathrm{SL}(\widetilde{M}^{(d)})$ and the genus and number of cusps of the resulting Teichmüller curve grow rapidly with d, as shown by Table 4.2

We will use the following rational directions: $(p_r, q_r) = (-(4r+1), 4r+3)$ for $0 \le r < d$. Observe that the matrix $\begin{pmatrix} 2r+1 & 2r \\ 4r+3 & 4r+1 \end{pmatrix}$ maps the direction (p_r, q_r) to $(-1, 0)$. Moreover, this matrix can be written as $T_{\mathrm{v}}^2 T_{\mathrm{h}}^{2r} T_{\mathrm{v}}$. By Figure 4.3, $T_{\mathrm{v}}^2 T_{\mathrm{h}}^{2r} T_{\mathrm{v}} \cdot M^{(d)} = N^{(d)}$, so this surface has only one horizontal cylinder.

The matrix T_{v} maps (p_r, q_r) to $(-(4r+1), 2)$. The surface $T_{\mathrm{v}} \cdot \widetilde{M}^{(d)}$, which we call $\widetilde{N}^{(d)}$, is a degree-eight cover of $T_{\mathrm{v}} \cdot M^{(d)} = N^{(d)}$, which we will now describe explicitly.

For each $g \in Q$, consider a copy $N_g^{(d)}$ of $N^{(d)}$. Each of these copies consists of d squares. We label the r-th bottom side of each square of $N_g^{(d)}$ with η_g^r and the left side of the leftmost square with ζ_g.

Let $m = (d+1)/2$, which satisfies $m = 2 \bmod 4$ since $d = 3 \bmod 8$. There are $m-1$ squares to the left and to the right of m in $N_g^{(d)}$. The labels of the top sides of the squares to the right of m are:

$$\eta_{gk}^{m-1}, \eta_{-g}^{m-2}, \eta_{-gk}^{m-3}, \eta_{g}^{m-4}, \ldots, \eta_{gk}^{5}, \eta_{-g}^{4}, \eta_{-gk}^{3}, \eta_{g}^{2}, \eta_{gk}^{1}.$$

The labels of the top sides of the squares to the left of m are:

$$\eta_{-gi}^{d}, \eta_{gj}^{d-1}, \eta_{gi}^{d-2}, \eta_{-gj}^{d-3}, \eta_{-gi}^{d-4}, \ldots, \eta_{gi}^{m+3}, \eta_{-gj}^{m+2}, \eta_{-gi}^{m+1}, \eta_{gj}^{m}.$$

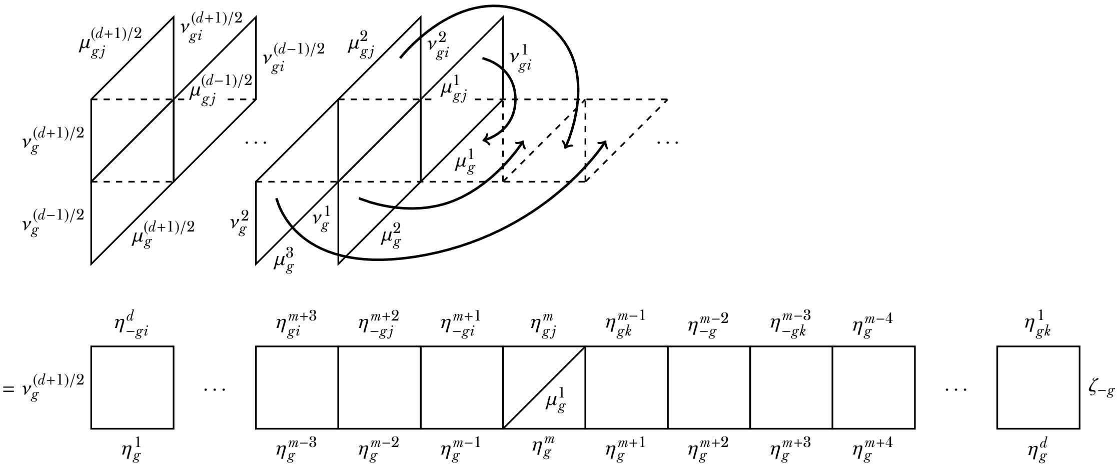


Figure 4.5: An illustration of $N_g^{(d)}$ and of the cut-and-paste operations used to obtain this description.

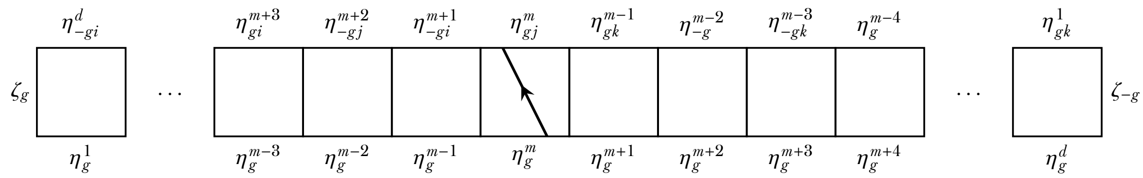


Figure 4.6: Direction $(-1, 2)$ on $N_g^{(d)}$.

In the two previous lists, the group elements in Q follow a 4-periodic pattern. Finally, we label the rightmost square of $N_g^{(d)}$ with ζ_{-g}. See Figure 4.5 for an illustration.

By a slight abuse of notation, from now on we will use the names η_g^r and ζ_g to refer to the elements of $H_1(\widetilde{N}^{(d)}, \Sigma; \mathbb{R})$ induced by the horizontal or vertical curves joining the two vertices of the side labelled η_g^r or ζ_g, oriented either rightwards or upwards. As for $\widetilde{M}^{(d)}$, we have that $\mathrm{Aut}(\widetilde{N}^{(d)}) \simeq Q$,. That is, we define an automorphism φ_g by mapping $N_h^{(d)}$ to $N_{gh}^{(d)}$ and these are the only automorphisms of $\widetilde{N}^{(d)}$ since $\mathrm{Aut}(N^{(d)})$ is trivial. The automorphism φ_{-1} induces a splitting $H_1(\widetilde{N}^{(d)}; \mathbb{R}) = H^+(\widetilde{N}^{(d)}) \oplus H^-(\widetilde{N}^{(d)})$. The space $H^-(\widetilde{N}^{(d)})$ is $4d$-dimensional and it is exactly the image of $H^-(\widetilde{M}^{(d)})$ by T_{v}. Let $\hat{\eta}_g^r = \eta_g^r - \eta_{-g}^r$ for $g \in Q^+ = \{1, i, j, k\}$ and $1 \leq r \leq d$. We have that each $\hat{\eta}_g^r$ is an absolute cycle since φ_{-1} fixes every singularity. Therefore, $\hat{\eta}_g^r \in H_1^-(\widetilde{N}^{(d)})$ and we obtain that $\{\hat{\eta}_g^r\}_{g \in Q^+, 1 \leq r \leq d}$ is a basis of $H_1^-(\widetilde{N}^{(d)})$.

Observe that $\widetilde{N}^{(d)}$ has four horizontal cylinders. Moreover, the matrix T_{h}^{2r} maps the direction $T_{\mathrm{v}}(p_r, q_r) = (-(4r+1), 2)$ to $(-1, 2)$. Therefore, understanding the direction (p_r, q_r) on $\widetilde{M}^{(d)}$ is equivalent to understanding the direction $(-1, 2)$ on $T_{\mathrm{h}}^{2r} \cdot \widetilde{N}^{(d)}$.

We will start the analysis for $r = 0$. For $g \in Q$, consider the trajectory induced by the direction $(-1, 2)$ on $N_g^{(d)}$ as in Figure 4.6. The resulting cylinder decomposition consists on eight cylinders. Indeed, observe that each cycle η_g^r is intersected twice by such trajectories. Therefore, the total number of intersections of all the η_g^r by all trajectories is $16d$. To obtain

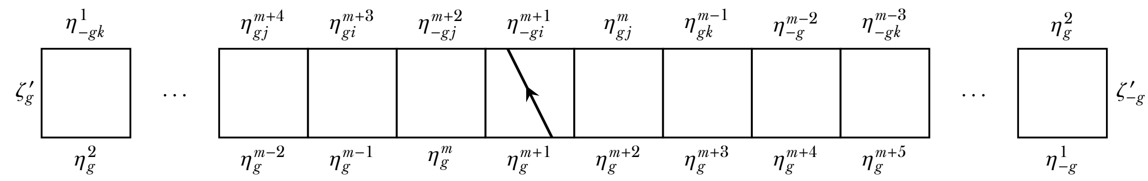


Figure 4.7: Direction $(-1, 2)$ on $T_{\mathrm{h}}^2 \cdot N_g^{(d)}$. The gluings are cyclically shifted and the signs of elements of Q on the labels $\eta^1_\bullet$ are changed.

that there are exactly eight cylinders in this decomposition, it is therefore enough to show that each trajectory intersects exactly $2d$ cycles η^r_g.

The trajectory in Figure 4.6 intersects the following cycles:

$$\begin{gathered}\eta^m_{g_1}, \eta^{m+1}_{g_2}, \eta^{m-1}_{g_3}, \eta^{m+2}_{g_4}, \eta^{m-2}_{g_5}, \dots, \eta^d_{g_{d-1}}, \eta^1_{g_d}, \zeta_{g_d}, \\ \eta^1_{g_{d+2}}, \eta^d_{g_{d+3}}, \dots, \eta^{m-2}_{g_{2d-3}}, \eta^{m+2}_{g_{2d-2}}, \eta^{m-1}_{g_{2d-1}}, \eta^{m+1}_{g_{2d}}, \eta^m_{g_{2d+1}}\end{gathered}$$

where the sequence $g_1, \dots, g_{2d+1}$ is obtained by (right-)multiplying g successively by

$$j, -i, k, -j, -1, i, -k \dots, j, 1, -i, k, \boxed{-1}, k, -i, 1, j, \dots, -k, i, -1, -j, k, -i, j. \tag{$*$}$$

The boxed -1 comes from the intersection with the vertical side labelled as ζ_{g_d}.

This sequence indeed describes a closed trajectory as $g_{2d+1} = g$. Indeed, the product can be computed from "inside out" by using that -1 is in the centre of Q. We obtain in this way that $g_{2d+1} = -g \cdot k^2 \cdot (-i)^2 \cdot 1^2 \cdot j^2 \cdots (-i)^2 j^2$. Moreover, the number of times 1^2 and $(-1)^2$ occur in this product is $(m-2)/2$, which is an even number as $m \bmod 4 = 2$, and the total number of terms is $d + 1$, which is also an even number. Thus, $g_{2d+1} = g$ and we conclude that the cylinder decomposition induced by $(-1, 2)$ has exactly eight cylinders. Moreover, we obtain that the action of $\mathrm{Aut}(\widetilde{N}^{(d)})$ on these waist curves is "well-behaved" in the sense of (2): naming the trajectory starting on $N_g^{(d)}$ as c^0_g, we get that $(\varphi_h)_* c^0_g = c^0_{hg}$.

Now, if $r = 1$ then $N_g^{(d)}$ is sheared horizontally in such a way that the labels η^{m+1}_g and η^{m+1}_{-gi} end up on the same square. We will consider this square to be the "middle" square and reglue the surface accordingly. The surface $T_{\mathrm{h}}^2 \cdot \widetilde{N}^{(d)}$ is the union of sheared and reglued versions of $N_g^{(d)}$, for $g \in Q$, that we call $T_{\mathrm{h}}^2 \cdot N_g^{(d)}$. See Figure 4.7 for an illustration.

In general, $T_{\mathrm{h}}^{2r} \cdot \widetilde{N}^{(d)}$, for $0 \le r < d$, is the surface obtained from $\widetilde{N}^{(d)}$ by cyclically shifting the labels on the top sides r times to the right, the ones on the bottom sides r times to the left, and changing the signs of the elements of Q of every label of the form $\eta^s_\bullet$ for $1 \le s \le r$. We conclude that the cylinder decomposition of $T_{\mathrm{h}}^{2r} \cdot \widetilde{N}^{(d)}$ induced by the direction $(-1, 2)$ consists of exactly eight cylinders in the same way as for the case $r = 0$ and denote their waist curves by c^r_g. The action of G is then well-behaved in the sense of (2). By construction, (1) also holds.

Let $\hat{c}^r_g = c^r_g - c^r_{-g}$ for $g \in Q^+$. It remains to prove (3), (4) and (5) to conclude the proof. We conjecture that these two conditions hold for every d belonging to the congruence class

$d = 3 \mod 8$.

Nevertheless, the previous discussion allows us to compute the intersection numbers explicitly using a computer. To this end, we use the versions of these conditions in terms of intersection numbers. To obtain (3) we can compute the intersection numbers $\langle \hat{c}_g^r, \hat{\eta}_h^s \rangle$ for each $0 \le r, s \le d$. Then, we can compute the determinant of the resulting matrix to show that it is not singular. This matrix also allows us to compute $\langle \hat{c}_g^r, \hat{c}_h^s \rangle$ by expressing each $\hat{c}_g^r$ in terms of the basis $\{\hat{\eta}_g^r\}_{g \in Q^+, 1 \le r \le d}$ to show (4) and (5). The computations were done for $11 \le d \le 299$. Observe that $d = 35$ and $d = 203$ are the only elements of $\{11, \ldots, 299\}$ satisfying $d = 3 \bmod 8$ and not belonging to $\mathcal{D}$, as $2 \times 35 = \binom{8}{4}$ and $2 \times 203 = \binom{29}{2}$. In this way, Theorem 4.1.1 is proved.

Conclusions

We will now briefly discuss how our work can be extended. We present this discussion in the same order as the chapters of the thesis.

The computation of the Rauzy–Veech and monodromy groups of strata of Abelian differentials is now completely solved. Nevertheless, the analogous question for the image inside the mapping class group, instead of the symplectic group, remains open. In this context, the work of Avila, Matheus and Yoccoz [AMY18] shows that the hyperelliptic modular Rauzy–Veech groups coincide with the orbifold fundamental groups of projectivised versions of the components. For genus three, such orbifold fundamental groups are explicitly known [LM14]. Moreover, the *geometric monodromy groups*, that is, the images of the orbifold fundamental groups inside the mapping class groups are known for higher genus by very recent work of Calderon [Cal19]. It would be interesting to study how these classifications are related to the Teichmüller geodesic flow. Thus, we formulate the following question:

Question 1. *To which extent can the geometric monodromy groups and the orbifold fundamental groups be recovered by the Teichmüller geodesic flow via the modular Rauzy–Veech groups* (*or similar groups for a different coding of the flow*)?

For the case of quadratic differentials, the simplicity of the Lyapunov spectra and the computation of the Rauzy–Veech and monodromy groups is only partially solved. The techniques we used for these partial proofs are very similar to the ones used for the Abelian case. While we conjecture that the Rauzy–Veech groups also coincide with the monodromy groups, we think that the proof is combinatorially very hard. Therefore, we think that some other coding scheme needs to be found in order to answer these questions with combinatorial methods. Such a coding, by *veering triangulations*, is being actively developed by the author and his coauthors. It is related to the work by Delecroix and Ulcigrai for hyperelliptic components [DU15].

Summarising, we formulate the following questions:

Question 2. *Are the Lyapunov spectra of typical quadratic differentials simple*?

Question 3. *What are the monodromy groups of the components of the strata of quadratic differentials and do they coincide with some groups arising from a coding of the Teichmüller geodesic flow, such as the Rauzy–Veech groups*?

For arbitrary affine invariant submanifolds, the monodromy groups of the Kontsevich–Zorich cocycle remain elusive. We have shown that some groups of the form $SO^*(2d)$ are realisable and we think that our techniques can be refined to obtain a positive-density set of $d \in \mathbb{N}$. Nevertheless, constructing every group of the form $SO^*(2d)$ will probably require some other methods. Moreover, Theorem 1.2.20 seems to suggest that, for the case of square-tiled surfaces, the only realisable groups are $Sp(2g)$, $SO^*(2d)$ and $SU(p, q)$, while the list of possible groups for general affine invariant submanifolds is larger by Theorem 1.2.13. It would be interesting to know the origin of this discrepancy. Thus, we formulate the following question:

Question 4. *Other than those coming from exceptional isomorphisms, are some exterior power representations of* $\mathfrak{su}(p, 1)$ *or some spin representations of* $\mathfrak{so}_{\mathbb{R}}(n)$ *realisable as the Lie algebra of the Zariski-closure of a factor of the monodromy group of a translation surface or of a square-tiled surface*?

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Appendix A

Base cases for Rauzy–Veech groups of minimal strata

In this appendix we will explicitly state some computations for the base cases of the induction used to prove Theorem 2.1.4. Specifically, these facts were used to show that some permutations belong to the desired minimal strata and that the modulo-two reduction of the Rauzy–Veech groups are the entirety of the orthogonal groups.

It is possible to find all elements of $\mathrm{NS}(\tau^{(d)})$ for small values of d by hand. This was done to find the base cases of the induction in Lemma 2.4.1. In particular:

Fact 1. $\mathrm{NS}(\tau^{(6)})$ *consists of 36 elements, which, written on the basis* $(\bar{e}_\alpha)_{\alpha=1}^{6}$, *are*:

$$\begin{array}{llllll}
(1,0,0,0,0,0), & (0,1,0,0,0,0), & (1,1,0,0,0,0), & (0,0,1,0,0,0), & (1,0,1,0,0,0), & (0,1,1,0,0,0) \\
(0,0,0,1,0,0), & (1,0,0,1,0,0), & (1,1,0,1,0,0), & (1,0,1,1,0,0), & (0,0,0,0,1,0), & (1,0,0,0,1,0) \\
(1,1,0,0,1,0), & (1,0,1,0,1,0), & (0,0,0,1,1,0), & (1,1,1,1,1,0), & (0,0,0,0,0,1), & (1,0,0,0,0,1) \\
(0,1,0,0,0,1), & (0,0,1,0,0,1), & (0,0,0,1,0,1), & (0,1,0,1,0,1), & (1,1,0,1,0,1), & (0,0,1,1,0,1) \\
(1,0,1,1,0,1), & (1,1,1,1,0,1), & (0,0,0,0,1,1), & (0,1,0,0,1,1), & (1,1,0,0,1,1), & (0,0,1,0,1,1) \\
(1,0,1,0,1,1), & (1,1,1,0,1,1), & (1,1,0,1,1,1), & (1,0,1,1,1,1), & (0,1,1,1,1,1), & (1,1,1,1,1,1).
\end{array}$$

Fact 2. $\mathrm{NS}(\sigma^{(8)})$ *consists of 120 elements, which, written on the basis* $(\bar{e}_\alpha)_{\alpha=1}^{8}$, *are*:

$$\begin{array}{llll}
(1,0,0,0,0,0,0,0), & (0,1,0,0,0,0,0,0), & (1,1,0,0,0,0,0,0), & (0,0,1,0,0,0,0,0) \\
(1,0,1,0,0,0,0,0), & (0,1,1,0,0,0,0,0), & (0,0,0,1,0,0,0,0), & (1,0,0,1,0,0,0,0) \\
(0,1,0,1,0,0,0,0), & (0,0,1,1,0,0,0,0), & (0,0,0,0,1,0,0,0), & (1,0,0,0,1,0,0,0) \\
(0,1,0,0,1,0,0,0), & (0,0,1,0,1,0,0,0), & (0,0,0,1,1,0,0,0), & (1,1,1,1,1,0,0,0) \\
(0,0,0,0,0,1,0,0), & (1,0,0,0,0,1,0,0), & (1,1,0,0,0,1,0,0), & (1,0,1,0,0,1,0,0) \\
(1,0,0,1,0,1,0,0), & (0,1,1,1,0,1,0,0), & (1,0,0,0,1,1,0,0), & (0,1,1,0,1,1,0,0) \\
(0,1,0,1,1,1,0,0), & (0,0,1,1,1,1,0,0), & (0,1,1,1,1,1,0,0), & (1,1,1,1,1,1,0,0) \\
(0,0,0,0,0,0,1,0), & (1,0,0,0,0,0,1,0), & (1,1,0,0,0,0,1,0), & (1,0,1,0,0,0,1,0) \\
(1,0,0,1,0,0,1,0), & (0,1,1,1,0,0,1,0), & (1,0,0,0,1,0,1,0), & (0,1,1,0,1,0,1,0) \\
(0,1,0,1,1,0,1,0), & (0,0,1,1,1,0,1,0), & (0,1,1,1,1,0,1,0), & (1,1,1,1,1,0,1,0) \\
(0,0,0,0,0,1,1,0), & (1,1,1,0,0,1,1,0), & (1,1,0,1,0,1,1,0), & (1,0,1,1,0,1,1,0)
\end{array}$$

$$
\begin{array}{llll}
(0,1,1,1,0,1,1,0), & (1,1,1,1,0,1,1,0), & (1,1,0,0,1,1,1,0), & (1,0,1,0,1,1,1,0) \\
(0,1,1,0,1,1,1,0), & (1,1,1,0,1,1,1,0), & (1,0,0,1,1,1,1,0), & (0,1,0,1,1,1,1,0) \\
(1,1,0,1,1,1,1,0), & (0,0,1,1,1,1,1,0), & (1,0,1,1,1,1,1,0), & (0,1,1,1,1,1,1,0) \\
(0,0,0,0,0,0,0,1), & (1,0,0,0,0,0,0,1), & (0,1,0,0,0,0,0,1), & (0,0,1,0,0,0,0,1) \\
(0,0,0,1,0,0,0,1), & (1,1,1,1,0,0,0,1), & (0,0,0,0,1,0,0,1), & (1,1,1,0,1,0,0,1) \\
(1,1,0,1,1,0,0,1), & (1,0,1,1,1,0,0,1), & (0,1,1,1,1,0,0,1), & (1,1,1,1,1,0,0,1) \\
(0,0,0,0,0,1,0,1), & (0,1,0,0,0,1,0,1), & (1,1,0,0,0,1,0,1), & (0,0,1,0,0,1,0,1) \\
(1,0,1,0,0,1,0,1), & (1,1,1,0,0,1,0,1), & (0,0,0,1,0,1,0,1), & (1,0,0,1,0,1,0,1) \\
(1,1,0,1,0,1,0,1), & (1,0,1,1,0,1,0,1), & (0,0,0,0,1,1,0,1), & (1,0,0,0,1,1,0,1) \\
(1,1,0,0,1,1,0,1), & (1,0,1,0,1,1,0,1), & (1,0,0,1,1,1,0,1), & (0,1,1,1,1,1,0,1) \\
(0,0,0,0,0,0,1,1), & (0,1,0,0,0,0,1,1), & (1,1,0,0,0,0,1,1), & (0,0,1,0,0,0,1,1) \\
(1,0,1,0,0,0,1,1), & (1,1,1,0,0,0,1,1), & (0,0,0,1,0,0,1,1), & (1,0,0,1,0,0,1,1) \\
(1,1,0,1,0,0,1,1), & (1,0,1,1,0,0,1,1), & (0,0,0,0,1,0,1,1), & (1,0,0,0,1,0,1,1) \\
(1,1,0,0,1,0,1,1), & (1,0,1,0,1,0,1,1), & (1,0,0,1,1,0,1,1), & (0,1,1,1,1,0,1,1) \\
(1,1,0,0,0,1,1,1), & (1,0,1,0,0,1,1,1), & (0,1,1,0,0,1,1,1), & (1,1,1,0,0,1,1,1) \\
(1,0,0,1,0,1,1,1), & (0,1,0,1,0,1,1,1), & (1,1,0,1,0,1,1,1), & (0,0,1,1,0,1,1,1) \\
(1,0,1,1,0,1,1,1), & (0,1,1,1,0,1,1,1), & (1,0,0,0,1,1,1,1), & (0,1,0,0,1,1,1,1) \\
(1,1,0,0,1,1,1,1), & (0,0,1,0,1,1,1,1), & (1,0,1,0,1,1,1,1), & (0,1,1,0,1,1,1,1) \\
(0,0,0,1,1,1,1,1), & (1,0,0,1,1,1,1,1), & (0,1,0,1,1,1,1,1), & (0,0,1,1,1,1,1,1).
\end{array}
$$

Proving that $\{\bar{e}_\alpha\}^{Q^{(d)}}_{\alpha\in\mathcal{A}} = \mathrm{NS}(\pi^{(d)})$ for the base cases of the induction can be done as follows: we start with the set $S_1 = \{\bar{e}_\alpha\}_{\alpha\in\mathcal{A}}$, which is contained in $\{\bar{e}_\alpha\}^{Q^{(d)}}_{\alpha\in\mathcal{A}}$. We define the set S_{k+1} as the union of S_k with the set of vectors of the form $v + \bar{e}_\alpha$, where $v \in S_k$ and $Q(v + \bar{e}_\alpha) = 1$. Clearly, $S_k \subseteq \{\bar{e}_\alpha\}^{Q^{(d)}}_{\alpha\in\mathcal{A}}$ for each k and, after a small number of iterations, it must be equal to $\{\bar{e}_\alpha\}^{Q^{(d)}}_{\alpha\in\mathcal{A}}$. We can then check that $\{\bar{e}_\alpha\}^{Q^{(d)}}_{\alpha\in\mathcal{A}} = \mathrm{NS}(\pi^{(d)})$.